Fourier Frame approximations: Algorithms and Applications

Roel Matthysen

May 23, 2016

Joint work with Daan Huybrechs
Department of Computer Science, KU Leuven
Goal
Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

Example: approximate $\exp(x)$ for increasing dof:

A set of Fourier basis functions restricted to a subdomain constitutes a Frame.
Goal
Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

Example: approximate $\exp(x)$ for increasing dof:

A set of Fourier basis functions restricted to a subdomain constitutes a Frame.
Goal
Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

Example: approximate $\exp(x)$ for increasing dof:

![Graphs showing Fourier interpolation and Fourier Frame](image)

(a) Fourier interpolation

(b) Fourier Frame

A set of Fourier basis functions restricted to a subdomain constitutes a Frame.
Goal
Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

Example: approximate $\exp(x)$ for increasing dof:

![Fourier interpolation and Fourier Frame graphs]

A set of Fourier basis functions restricted to a subdomain constitutes a Frame.
**Goal**

Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

**Example:** approximate $\exp(x)$ for increasing dof:

(A) Fourier interpolation

(B) Fourier Frame

---

*A set of Fourier basis functions restricted to a subdomain constitutes a Frame.*
Goal
Given a function $f$ as $N$ data points on an equispaced grid, find an expansion in Fourier basis functions that

- converges fast
- is easily computable ($O(N)$ operations).

Example: approximate $\exp(x)$ for increasing dof:

A set of Fourier basis functions restricted to a subdomain constitutes a Frame.
Fourier Frame on $[-T, T]$

- Basis functions $\phi_m(x) = \frac{1}{\sqrt{2T}} e^{i \frac{\pi m x}{T}}$, $m = -M, \ldots, M$
- Approximation $g(x) = \sum_{k=-M}^{M} a_k \phi_k(x)$

Fitting the data on $[-1, 1]$

- Collocation at equidistant points $x_j = \frac{j}{N}, j = -N, \ldots, N$.
- Least squares problem $A_{ij} = \phi_i(x_j), b_j = f(x_j)$
  
  \[ A a = b \]

- $A$ is DFT-subblock
- Some oversampling $N = \eta M$ required, $A$ is a tall matrix ($\eta \sim 2$)
- Fast solver for $A a = b$ needed
- Fast algorithm exists for $T = 2$ (Lyon 2011)
A closer look at $A$

Singular values of $A$

- Spectrum has a plateau shape (Slepian 1978, Wilson 1987)
- Size of transition/plunge/problematic region
  
  $1 - \epsilon > \theta_i > \epsilon$

  grows as $O(\log N)$ for 1D problems.
- System matrix is extremely ill-conditioned
Ill-conditioning explained

Fourier series corresponding to the right singular vectors of $A$:

- Solving with Truncated SVD yields approximation to machine accuracy (Adcock, Huybrechs, Martin-Vaquero 2014)
Solving the least-squares system

Truncated SVD

\[ A = U \Sigma V' \]
\[ x = V \Sigma^{-1} U' b \]
\[ x = V_1 \Sigma_1^{-1} U_1' b_1 + V_{mid} \Sigma_{mid}^{-1} U_{mid}' b_{mid} + V_{\epsilon} \Sigma_{\epsilon}^{-1} U_{\epsilon}' b_{\epsilon} \]

Assumption: \( b_{\epsilon} \) is negligible (discrete Picard condition)

\( x_1 \) is easy when \( x_{mid} \) is known

\[ b_1 = b - b_{mid} = b - Ax_{mid} \]
\[ x_1 = V_1 \Sigma_1^{-1} U_1' b_1 \approx V_1 \Sigma_1 U_1' b_1 = A' b_1 \]

All you need is one application of \( A \) and \( A' \) (fast!)

How to find \( x_{mid} \)?
**Isolating the problematic singular values**

- ‘Projection operator’ $I - AA'$ isolates the middle singular values

\[
(I - AA')A = U(\Sigma - \Sigma^3)V'
\]

The numerical nullspace of $(I - AA')A$ contains both the 1 and $\epsilon$-regions.

- Solving $(I - AA')Ax = (I - AA')b$ yields $x_{mid}$
- $(I - AA')A$ has numerical rank $O(\log N)$
- This can be solved efficiently in $O(N \log^{2} N)$
Algorithm

\[(I - AA')Ax_\beta = (I - AA')b\]
\[x_\alpha = A'(b - Ax_\beta)\]
\[x = x_\alpha + x_\beta\]

Results

(a) timings

(b) convergence
Fourier Frame

• A consists of selected rows of the 2D DFT matrix.

• Partially proven: the problematic region grows as $O(\sqrt{N})$ for $N$ total points.
• The algorithm complexity becomes $O(N^2)$
Experimental results for 2D extensions:

- Complexity is $O(N^2)$, as expected
- Convergence speed is seemingly conserved
Fourier Frame approximations: Algorithms and Applications

Roel Matthysen

2D Fourier Frame Approximation

\[ f(x, y) = \cos(20x + 22y) \]
Observation: You can add extra rows and/or columns to $A$ while preserving the plateau shape. Let $\bar{A} = [A \ b]$. Then because of the interlacing property of singular values

$$\bar{\sigma}_1 > \sigma_1 > \bar{\sigma}_2 > \ldots$$

the spectra are similar:

$$sp(A) : 1 > \sigma_1 > \cdots > \sigma_k > (1 - \epsilon) > \sigma_{k+1} > \cdots > \sigma_l > \epsilon > \sigma_{l+1} > \cdots > 0.$$  
$$sp(\bar{A}) : 1 > \bar{\sigma}_2 > \cdots > \bar{\sigma}_k > (1 - \epsilon) > \bar{\sigma}_{k+2} > \cdots > \bar{\sigma}_l > \epsilon > \bar{\sigma}_{l+2} > \cdots > 0.$$  

Problematic region contains a maximum of 3 extra singular values. This might be useful for

- Some extra functions added to the frame
- Boundary conditions when solving differential equations
Fourier Frame approximations: Algorithms and Applications

Roel Matthysen

Differential Equations

Derivative is diagonal operator

\[ f(x) = \sum_{m=-M}^{M} a_m \phi_m(x), \quad f'(x) = \sum_{m=-M}^{M} \frac{a_m i \pi m}{T} \phi_m(x) \]

A differential equation

\[ \Delta A + k^2 A = f \]

becomes

\[ (D_x^2 + D_y^2 + k^2 I)c_A = c_f \]

Implementation

- Boundary conditions are sampled at the boundary and added as extra rows in the system matrix. Complexity remains \( O(N \log N) / O(N^2) \).
- If the diagonal operator is not invertible, care has to be taken of the DC coefficient.
Differential Equations

Poisson Equation

\[ \Delta A = \cos(2\pi(x + y)), \quad \delta A(x, y)/\delta y = 0, \quad (x, y) \in \delta \Omega \]
Differential Equations

Laplace Equation

\[ \Delta A = 0, \quad A(x, y) = x - y, \quad (x, y) \in \delta \Omega \]
Differential Equations

Helmholtz Equation

\[ \Delta A + 35^2 A = e^{-200((x-0.3)^2 + (y+0.3)^2)} \]

\[ A(x, y) = 0, \quad (x, y) \in \delta \Omega \]
Fourier Frame approximations: Algorithms and Applications

Roel Matthysen

A closer look at $A$

Solving the system

2D Frames

Possible extensions

GitHub Julia

Julia code available at https://github.com/daanhb/Framefuns.jl