Fourier Extension and Prolate Spheroidal Wave Theory: Fast algorithms

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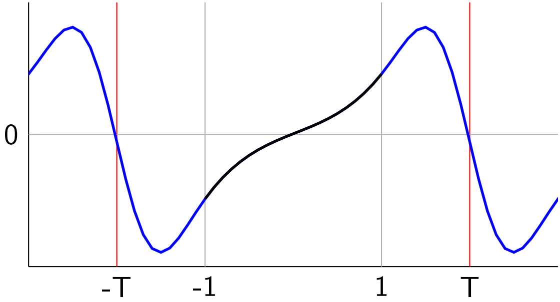
Introduction

Fourier Extension

Function f given on $[-1,1]$, construct Fourier series on larger domain $[-T,T]$.

$$a := \arg \min_{a\in\mathbb{R}^{2N+1}} ||f - \sum_{n=-N}^{N} a_n e^{i\frac{\pi n}{T}x}||_{L^2_{[-1,1]}}$$



Fourier Extension

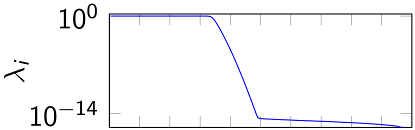
- Formulation as a Least Squares problem:

$$Aa = b$$

$$\begin{bmatrix} \ddots & \vdots & \\ & \int_{-1}^{1} \phi_k(x)\phi_l(x)dx & \\ & \vdots & \ddots \end{bmatrix} a = \begin{bmatrix} \vdots \\ \int_{-1}^{1} f(x)\phi_k(x)dx \\ \vdots \end{bmatrix}$$

$$\phi_k(x) = e^{i\frac{\pi k}{T}x}, \quad k = -N, \dots, N$$

- A is a subblock of the prolate matrix



- The exact solution of the LS problem is unbounded with N, but small norm solutions (TSVD) exist.

Introduction

Setting : Equispaced grid

Samples $f(x_l)$ given, where $x_l = l/M$, $l = -M, \ldots, M$.

$$a := \arg \min_{a \in \mathbb{R}^{2N+1}} \sum_{l=-M}^{M} \left(f(x_l) - \sum_{n=-N}^{N} a_n e^{i \frac{\pi n}{T} x_l} \right)^2 .$$

Linear Algebra problem

- Solve, in a least squares sense,

$$Aa \approx b, \qquad A_{kl} = e^{i \frac{\pi k}{T} x_l}, \qquad b_l = f(x_l)$$

- Normal equations $A'Aa = A'b$ worsen ill-conditioning
- Convergence to machine precision ϵ proven for TSVD
- Fast algorithms needed.

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PSWFs (Slepian, Landau, Pollak)

Given Fourier transform of $f(x)$ in $\mathcal{L}^2_{[-\infty,\infty]}$,

$$\mathcal{F}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} ds,$$

Define the time- and bandlimiting operators as

$$\mathcal{D}f(x) = \begin{cases} f(x) & |x| \leq T \\ 0 & |x| > T \end{cases} \qquad \mathcal{B}f(x) = \int_{-\Omega}^{\Omega} \mathcal{F}(\xi) e^{i2\pi\xi x} d\xi,$$

Then the PSWFs are the eigenfunctions of the operator $\mathcal{BD}$.

$$\lambda_i \psi_i(x) = \mathcal{BD}\psi_i(x)$$

$$\lambda_i \psi_i(x) = \int_{-T}^{T} \frac{\sin(2\pi\Omega(x-s))}{\pi(x-s)} \psi_i(s) ds,$$

$1 > \lambda_0 > \lambda_1 > \cdots > 0.$

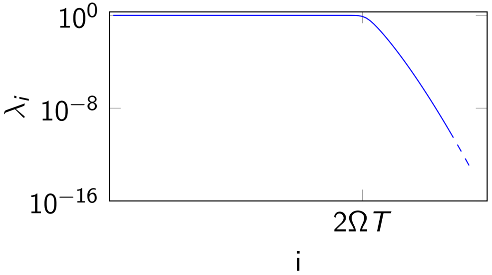
Concentration problem

$$\lambda_i \psi_i(x) = \mathcal{B}\mathcal{D}\psi_i(x)$$

PSWFs answer the question: *"What is the maximum concentration of a bandlimited function inside a given interval?"*

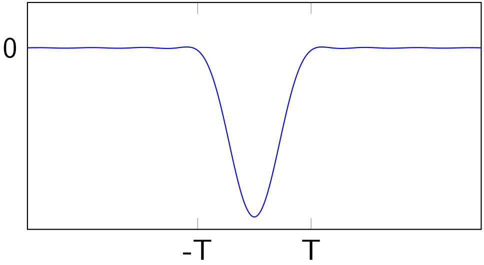
$$\frac{\int_{-T}^{T} \psi_i(x)\psi_i(x)dx}{\int_{-\infty}^{\infty} \psi_i(x)\psi_i(x)dx} = \lambda_i$$

Exponential decay sets in after $\sim 2\Omega T$ eigenvalues.



Properties

PSWF $\psi_0(x)$, $2\Omega T \approx 4$

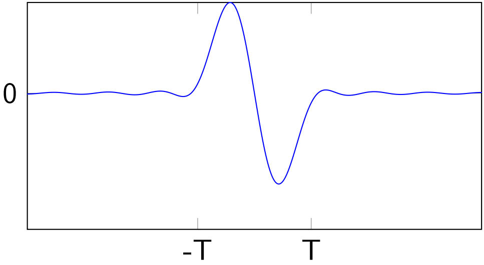


Properties

- The ψ_i are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- ψ_i has i zeros inside $[-T, T]$
- ψ_i is even and odd with i

Properties

PSWF $\psi_1(x)$, $2\Omega T \approx 4$



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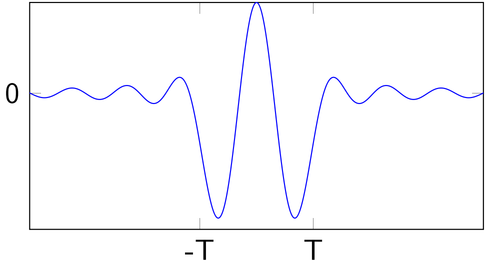
PSWF $\psi_2(x)$, $2\Omega T \approx 4$



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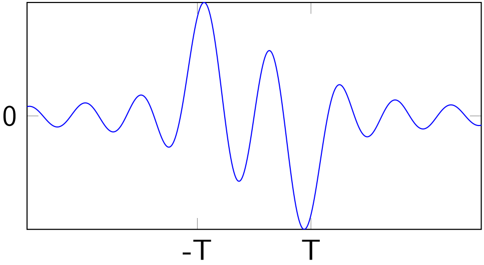
Properties

PSWF $\psi_3(x)$, $2\Omega T \approx 4$



Properties

- The ψ_i are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
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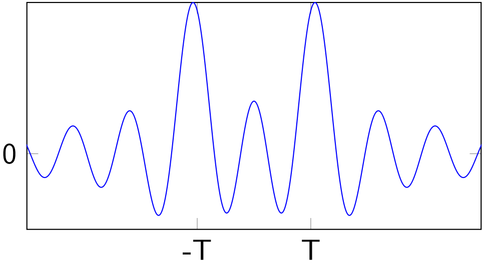
Properties

PSWF $\psi_4(x)$, $2\Omega T \approx 4$



Properties

- The ψ_i are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- ψ_i has i zeros inside $[-T, T]$
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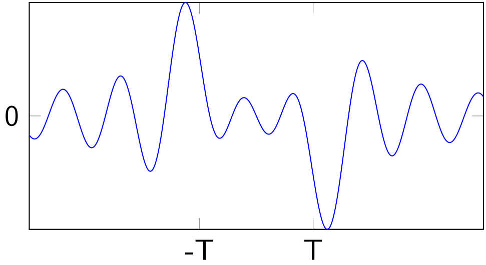
Properties

PSWF $\psi_5(x)$, $2\Omega T \approx 4$



Properties

- The ψ_i are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
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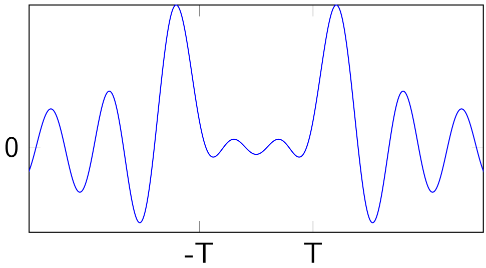
Properties

PSWF $\psi_6(x)$, $2\Omega T \approx 4$



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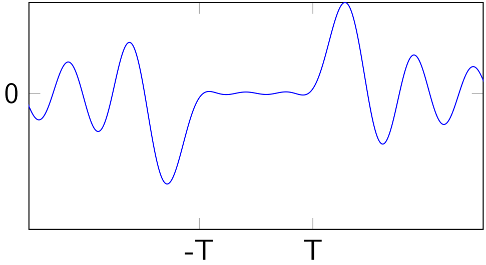
PSWF $\psi_7(x)$, $2\Omega T \approx 4$



Properties

- The ψ_i are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- ψ_i has i zeros inside $[-T, T]$
- ψ_i is even and odd with i

Other interesting properties

Spectrum localisation

The ψ_i are eigenfunctions of the finite Fourier transform,

$$\int_{-\Omega}^{\Omega} e^{i2\pi s\xi}\psi_i(s)ds = \alpha_i\psi_i(\xi).$$

Commutation with 2nd order differential operator

the differential operator

$$P_x = \left(1 - \frac{x^2}{T^2}\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} - (2\pi\Omega T)^2 x^2$$

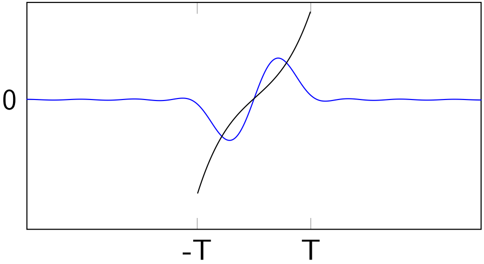
commutes with $\mathcal{DB}$, i. e. for any bandlimited f

$$P\mathcal{DB}f = \mathcal{DB}Pf, \qquad \text{and} \quad P_x\psi_i(x) = \chi_i\psi_i(x).$$

Bandlimited approximation

- Problem : *"find bandlimited $\tilde{f}$ so that $\tilde{f}$ agrees with f on the interval $[-T, T]$"*
- Solution : expand in PSWFs

$$\tilde{f} = \sum_k \langle f, \psi_i \rangle \psi_i$$



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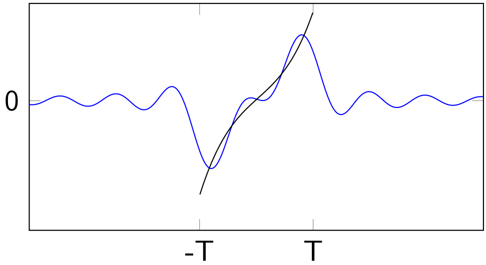

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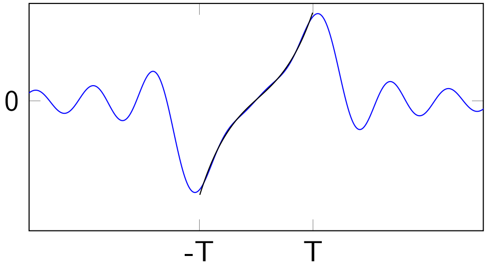

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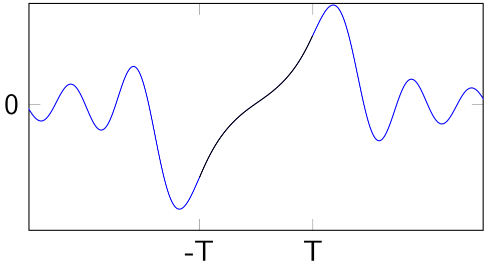


Discrete PSWFs

Given a function $g(x)$ on $[-T, T]$, and it's DTFT

$$g(x) = \sum_{n=-\infty}^{\infty} \mathcal{G}[n] e^{-i\frac{\pi n}{T}x} \qquad \mathcal{G}[n] = \frac{1}{2\pi}\int_{-T}^{T} g(x) e^{i\frac{\pi n}{T}x} dx,$$

Now redefine the bandlimiting operators as

$$\mathcal{D}g(x) = \begin{cases} g(x) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \qquad \mathcal{B}g(x) = \sum_{n=-N}^{N} \mathcal{G}[n] e^{-i\frac{\pi n}{T}x}$$

Then the discrete PSWFs are again the eigenfunctions of the operator $\mathcal{BD}$.

$$\lambda_i \psi_i(x) = \mathcal{BD}\psi_i(x)$$

Properties

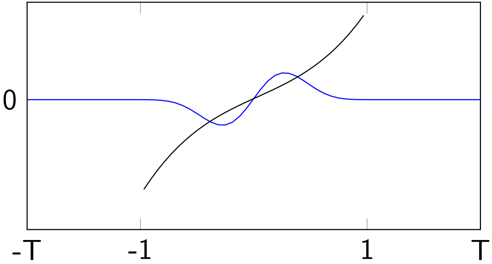
Write $\psi_i(x)$ as

$$\psi_i(x) = \sum_{n=-N}^{N} v_i[n] e^{-i\frac{\pi n}{T}x},$$

- $\mathcal{D}\psi_i(x) = \lambda_i \sum_{n=-\infty}^{\infty} v_i[n] e^{-i\frac{\pi n}{T}x}$
- Both $\psi_i(x)$ and $v_i[n]$ have similar properties to PSWFs (double orthogonality, even/odd, zeros)
- There are $2N+1$ nonzero eigenvalues, with exponential decay starting from $\lambda_{\frac{2N+1}{T}}$.
- Both the $\psi_i(x)$ and $v_i[n]$ commute with a second order differential operator.

Connection to FE

Bandlimited Extrapolation



Eigenvalue problem

Writing out $\lambda_i \psi_i(x) = \mathcal{B}\mathcal{D}\psi_i(x)$ for the Fourier coefficients v_i leads to $Av_i = \lambda_i v_i$, where

$$A_{ij} = \int_{-1}^{1} e^{i\frac{\pi(i-j)}{T}x} dx,$$

which is the matrix of the continuous problem.

Connection to FE

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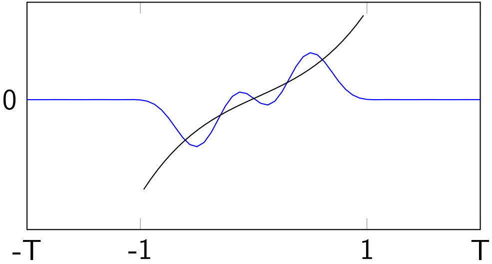

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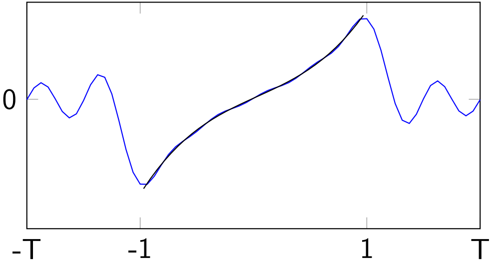
Connection to FE

Bandlimited Extrapolation



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which is the matrix of the continuous problem.

P-DPSSs

Given the DFT for a sequence of length $2L+1$,

$$H_k = \sum_{n=-L}^{L} h[n] e^{-i\frac{2\pi kn}{2L+1}} \qquad h[n] = \frac{1}{2L+1} \sum_{k=-L}^{L} H_k e^{i\frac{2\pi kn}{2L+1}}.$$

define the discrete time- and bandlimiting operator as

$$\mathcal{D}h = \begin{cases} h[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases} = Dh$$

$$\mathcal{B}h = \frac{1}{2L+1} \sum_{k=-N}^{N} H_k e^{\frac{i2\pi kn}{2L+1}} = Bh.$$

Then the P-DPSSs are the eigenvectors of

$$\lambda_i \phi_{N,M,i} = BD\phi_{N,M,i}.$$

Properties

- $\lambda_i \phi_{N,M,i}$ and $\mathcal{D}\phi_{N,M,i}$ are DFT pairs.
- $\sum_{k=-M}^{M} \phi_{N,M,i}[k]^2 = \lambda_i$
- Eigenvalues decay exponentially after $\sim (2N+1)T$ eigenvalues.
- $\phi_{N,M,i}$ satisfies a second order difference equation.
- The matrix DBD is equal to the normal matrix $A'A$ of the discrete Fourier Extension problem
- The difference operator T_n commutes with $A'A$, easy computation of $\phi_{M,N,i}$
- The left- and right singular vectors of A are given by $\phi_{N,M,i}$ and $\phi_{M,N,i}$ respectively.

Summary

For any prolate type object, we have:

- prolate type object is eigenfunction of $\mathcal{BD}$
- Time localisation proportional to exponential decaying λ_i
- Double orthogonality
- Frequency transform is another prolate type object
- $\mathcal{BD}$ commutes with a second order differential operator

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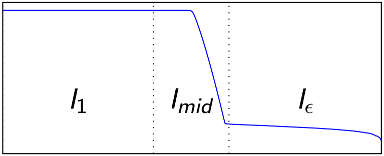
Fast Algorithms

Singular Values

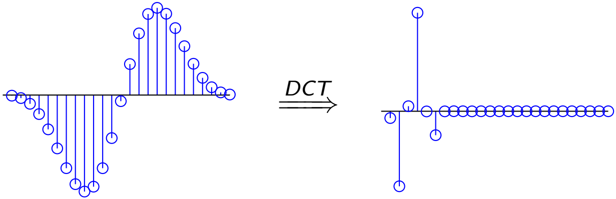


General Principle

- Isolate and solve for the middle part, at $O(\log N)$ cost.
- Exploit the good condition of the first part.
- Truncate the last part.

Exploiting frequency properties

- Frequency localization of singular vectors (left & right), proportional to corresponding singular value



- $A = USV^*$, C=DCT-matrix

$$CU \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix}$$

$$CA \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix} \begin{bmatrix} S_1 V_1^* \\ S_2 V_2^* \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}$$

Exploiting frequency properties

DCT result

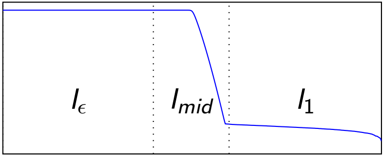
- $CA = \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}$, $Cb = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \end{bmatrix}$, where
  - $\kappa(\overline{A}_1) \approx 1$, full rank
  - $\kappa(\overline{A}_2) \approx \epsilon^{-1}$, $rank(\overline{A}_2) = O(\log N)$

Algorithm using random matrices

1. Solve $\overline{A}_1 a_1 = \overline{b}_1$ iteratively
2. Solve $\overline{A}_1 c = \overline{A}_1 r$, for a number of random vectors r
   - $r - c$ is in $null(\overline{A}_1)$
3. Construct orthogonal basis for $\{\overline{A}_2(r_i - c_i)\}$
4. Solve $\overline{A}_2 a_2 = \overline{b}_2 - \overline{A}_2 a_1$ with $a_2 \in null(\overline{A}_1)$
5. $a = a_1 + a_2$

Exploiting commuting operator

- Both left and right singular vectors of A are P-DPSS
- Split eigenvalues of A.



- Splitted SVD, $\Sigma_1 \approx I$ and $\Sigma_\epsilon \approx \mathbf{0}$

$$A = \begin{bmatrix} U_1 & U_{mid} & U_\epsilon \end{bmatrix} \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_{mid} & \\ & & \Sigma_\epsilon \end{bmatrix} \begin{bmatrix} V_1 & V_{mid} & V_\epsilon \end{bmatrix}'$$

$$\approx U_1 V_1' + U_{mid} \Sigma_{mid} V_{mid}'$$

Exploiting commuting operator

Orthogonal solutions

Split solution a into distinct parts $a_1 \in \text{span}\{V_1\}$, $a_{mid} \in \text{span}\{V_{mid}\}$, so that:

$$U_1 \Sigma_1 V_1' a_1 = b_1, \qquad U_{mid} \Sigma_{mid} V_{mid}' a_{mid} = b_{mid}.$$

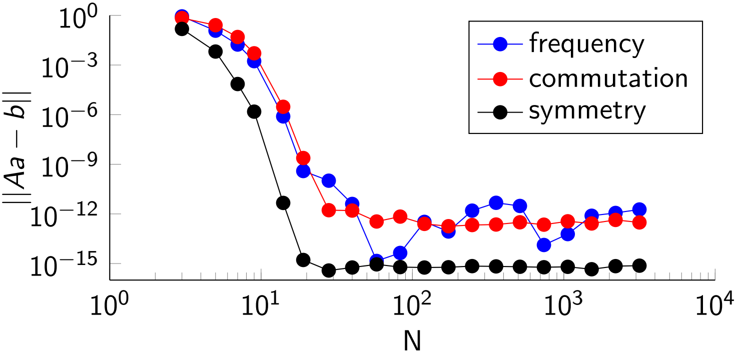
The trick here is that for b_1, $a_1 = V_1 \Sigma_1^{-1} U_1' b_1 = A' b_1$.

Algorithm

1. Find U_{mid}, V_{mid}, Σ_{mid} using the tridiag. matrix
2. $a_{mid} = V_{mid} \Sigma_{mid}^{-1} U_{mid}^T b$
3. $b_1 = b - A a_{mid}$
4. $a_1 = A' b_1$
5. $a = a_1 + a_{mid}$

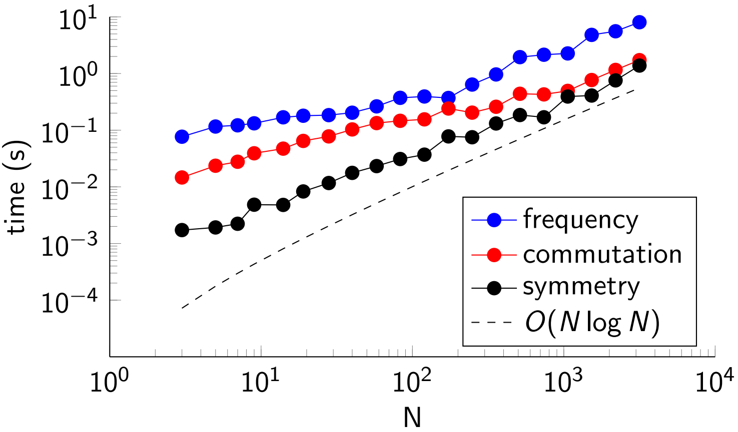
Numerical Results

Accuracy



- Accuracy overall is a bit worse than the symmetry-exploiting algorithm by M. Lyon
- Convergence seems to start slightly slower

Runtime



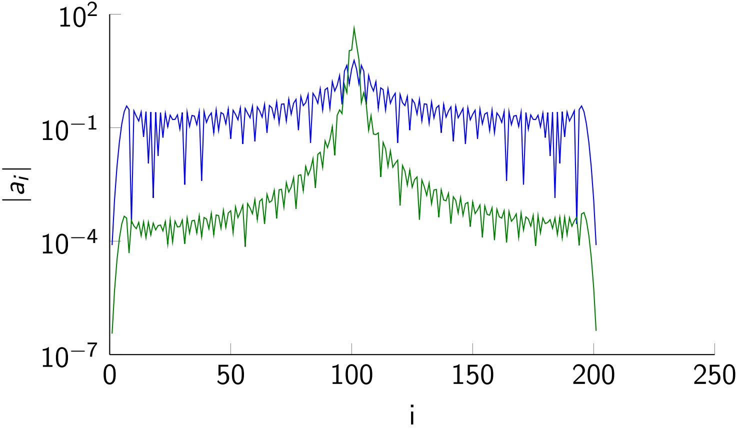
- All algorithms are $O(N \log(N)^2)$
- At least for large N, symmetric and commutation are close

Smooth functions

Sobolev smoothing of the coefficients by solving

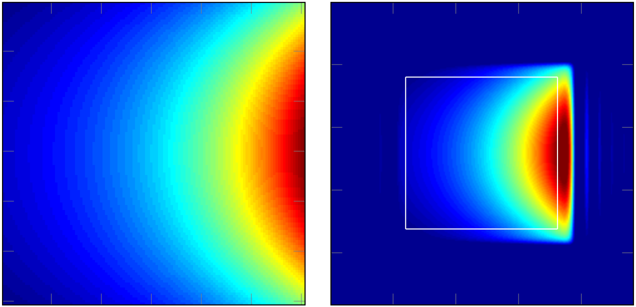
$$Da \approx Db \quad \text{s.t.} \quad V'_{mid}a = 0.$$

using QR orthogonalization.



Easy 2D

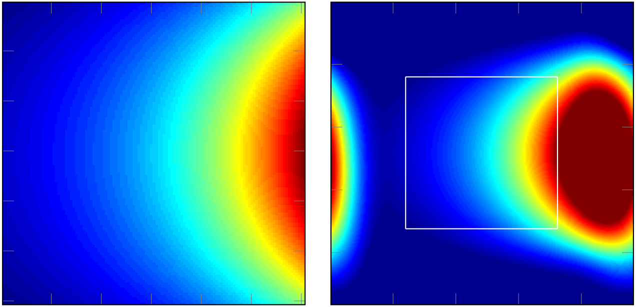
When extending a function on a rectangle, tensor-product structure can be exploited.



This algorithm has complexity $O(N \log N)$ for a total of N points.

Easy 2D - Smooth solutions

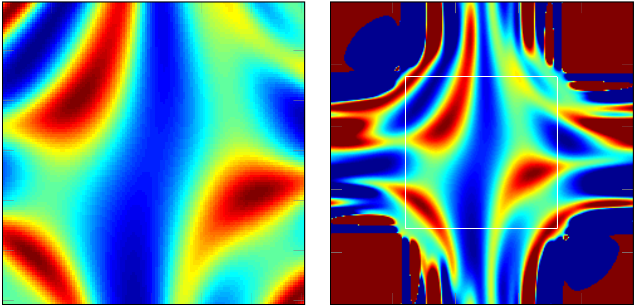
The 1D smoothing can be applied to obtain a smooth extension



- Accurate and smooth solution for well-behaved functions

Easy 2D - Smooth solutions (bis)

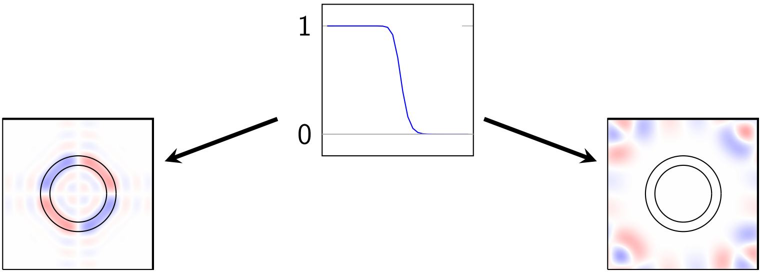
The 1D smoothing can be applied to obtain a smooth extension



- For more difficult functions, smoothness is limited to the borders.

Difficult 2D - General Domains

PSWFs generalise (at least partially) to any domain



- Is there symmetry to exploit?
- Does a commuting operator exist outside of tensor-product domains?
- What about frequency domain localisation?