Fourier Extension and Prolate Spheroidal Wave Theory: Fast algorithms

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Fourier Extension

Function $f$ given on $[-1, 1]$, construct Fourier series on larger domain $[-T, T]$.

$$a := \arg \min_{a \in \mathbb{R}^{2N+1}} \| f - \sum_{n=-N}^{N} a_n e^{i \frac{\pi n x}{T}} \|_{L^2[-1,1]}$$
Fourier Extension

- Formulation as a Least Squares problem:

\[ Aa = b \]

\[
\begin{pmatrix}
\vdots & \vdots \\
\int_{-1}^{1} \phi_k(x) \phi_l(x) dx & \cdot \\
\vdots & \cdot \\
\end{pmatrix}
\begin{pmatrix}
\cdot \\
\cdot \\
\end{pmatrix}
\begin{pmatrix}
\int_{-1}^{1} f(x) \phi_k(x) dx & \cdot \\
\cdot & \cdot \\
\end{pmatrix}
\]

\[ \phi_k(x) = e^{i \frac{\pi k}{T} x}, \quad k = -N, \ldots, N \]

- \( A \) is a subblock of the prolate matrix

\[ 10^0 \]

\[ 10^{-14} \]

- The exact solution of the LS problem is unbounded with \( N \), but small norm solutions (TSVD) exist.
Introduction

Setting: Equispaced grid

Samples \( f(x_l) \) given, where \( x_l = l/M, \ l = -M, \ldots, M \).

\[
a : = \arg \min_{a \in \mathbb{R}^{2N+1}} \sum_{l=-M}^{M} \left( f(x_l) - \sum_{n=-N}^{N} a_n e^{i \frac{\pi n}{T} x_l} \right)^2.
\]

Linear Algebra problem

- Solve, in a least squares sense,

\[
Aa \approx b, \quad A_{kl} = e^{i \frac{\pi k}{T} x_l}, \quad b_l = f(x_l)
\]

- Normal equations \( A' A a = A' b \) worsen ill-conditioning
- Convergence to machine precision \( \epsilon \) proven for TSVD
- Fast algorithms needed.
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PSWFs (Slepian, Landau, Pollak)

Given Fourier transform of $f(x)$ in $L^2_{[-\infty, \infty]}$,

$$\mathcal{F}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} ds,$$

Define the time- and bandlimiting operators as

$$Df(x) = \begin{cases} f(x) & |x| \leq T \\ 0 & |x| > T \end{cases} \quad Bf(x) = \int_{-\Omega}^{\Omega} \mathcal{F}(\xi) e^{i2\pi \xi x} d\xi,$$

Then the PSWFs are the eigenfunctions of the operator $BD$.

$$\lambda_i \psi_i(x) = BD \psi_i(x)$$

$$\lambda_i \psi_i(x) = \int_{-T}^{T} \frac{\sin(2\pi \Omega(x - s))}{\pi(x - s)} \psi_i(s) ds,$$

$$1 > \lambda_0 > \lambda_1 > \cdots > 0.$$
Concentration problem

$$\lambda_i \psi_i(x) = B D \psi_i(x)$$

PSWFs answer the question: "What is the maximum concentration of a bandlimited function inside a given interval?"

$$\frac{\int_{-T}^{T} \psi_i(x) \psi_i(x) dx}{\int_{-\infty}^{\infty} \psi_i(x) \psi_i(x) dx} = \lambda_i$$

Exponential decay sets in after $\sim 2\Omega T$ eigenvalues.
Properties

PSWF $\psi_0(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
Properties

PSWF $\psi_1(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$

Properties
Properties

PSWF $\psi_2(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$.
- $\psi_i$ has $i$ zeros inside $[-T, T]$.
- $\psi_i$ is even and odd with $i$.
Properties

PSWF $\psi_3(x), 2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
Properties

PSWF $\psi_4(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
Properties

PSWF $\psi_5(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
PSWF $\psi_6(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
Properties

PSWF $\psi_7(x)$, $2\Omega T \approx 4$

- The $\psi_i$ are orthogonal on both $[-T, T]$ and $[-\infty, \infty]$
- $\psi_i$ has $i$ zeros inside $[-T, T]$
- $\psi_i$ is even and odd with $i$
Other interesting properties

Spectrum localisation

The $\psi_i$ are eigenfunctions of the finite Fourier transform,

$$\int_{-\Omega}^{\Omega} e^{i2\pi s\xi} \psi_i(s) ds = \alpha_i \psi_i(\xi).$$

Commutation with 2nd order differential operator

the differential operator

$$P_x = \left(1 - \frac{x^2}{T^2}\right) \frac{d^2}{dx^2} - 2x \frac{d}{dx} - (2\pi\Omega T)^2 x^2$$

commutes with $\mathcal{D}B$, i.e. for any bandlimited $f$

$$PDBf = DBPf, \quad \text{and} \quad P_x \psi_i(x) = \chi_i \psi_i(x).$$
Bandlimited approximation

- **Problem**: “find bandlimited \( \tilde{f} \) so that \( \tilde{f} \) agrees with \( f \) on the interval \([-T, T]\)”
- **Solution**: expand in PSWFs

\[
\tilde{f} = \sum_k \langle f, \psi_i \rangle \psi_i
\]
Bandlimited approximation

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Bandlimited approximation

- Problem: “find bandlimited $\tilde{f}$ so that $\tilde{f}$ agrees with $f$ on the interval $[-T, T]$”

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$$\tilde{f} = \sum_k \langle f, \psi_i \rangle \psi_i$$

![Graph showing bandlimited approximation](image-url)
Bandlimited approximation

- Problem: “find bandlimited $\tilde{f}$ so that $\tilde{f}$ agrees with $f$ on the interval $[-T, T]$”
- Solution: expand in PSWFs

$$\tilde{f} = \sum_{k} \langle f, \psi_i \rangle \psi_i$$
Discrete PSWFs

Given a function $g(x)$ on $[-T, T]$, and it's DTFT

$$g(x) = \sum_{n=-\infty}^{\infty} G[n] e^{-i \frac{\pi n}{T} x} \quad G[n] = \frac{1}{2\pi} \int_{-T}^{T} g(x) e^{i \frac{\pi n}{T} x} dx,$$

Now redefine the bandlimiting operators as

$$Dg(x) = \begin{cases} g(x) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad Bg(x) = \sum_{n=-N}^{N} G[n] e^{-i \frac{\pi n}{T} x}$$

Then the discrete PSWFs are again the eigenfunctions of the operator $BD$.

$$\lambda_i \psi_i(x) = BD \psi_i(x)$$
Write $\psi_i(x)$ as

$$
\psi_i(x) = \sum_{n=-N}^{N} v_i[n] e^{-i \frac{\pi n}{T} x},
$$

- $D\psi_i(x) = \lambda_i \sum_{n=-\infty}^{\infty} v_i[n] e^{-i \frac{\pi n}{T} x}$
- Both $\psi_i(x)$ and $v_i[n]$ have similar properties to PSWFs (double orthogonality, even/odd, zeros)
- There are $2N + 1$ nonzero eigenvalues, with exponential decay starting from $\lambda \frac{2N+1}{T}$
- Both the $\psi_i(x)$ and $v_i[n]$ commute with a second order differential operator.
Connection to FE

Bandlimited Extrapolation

Eigenvalue problem

Writing out \( \lambda_i \psi_i(x) = BD \psi_i(x) \) for the Fourier coefficients \( v_i \)
leads to \( Av_i = \lambda_i v_i \), where

\[
A_{ij} = \int_{-1}^{1} e^{i \frac{\pi (i-j)}{T} x} dx,
\]

which is the matrix of the continuous problem.
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Connection to FE

Bandlimited Extrapolation

Eigenvalue problem

Writing out \( \lambda_i \psi_i(x) = BD \psi_i(x) \) for the Fourier coefficients \( v_i \) leads to \( A v_i = \lambda_i v_i \), where

\[
A_{ij} = \int_{-1}^{1} e^{i \frac{\pi (i-j)}{T} x} dx,
\]

which is the matrix of the continuous problem.
Writing out $\lambda_i \psi_i(x) = B \mathcal{D} \psi_i(x)$ for the Fourier coefficients $v_i$ leads to $A v_i = \lambda_i v_i$, where

$$A_{ij} = \int_{-1}^{1} e^{i \pi (i-j) x/T} dx,$$

which is the matrix of the continuous problem.
P-DPSSs

Given the DFT for a sequence of length $2L + 1$,

$$H_k = \sum_{n=-L}^{L} h[n] e^{-i \frac{2\pi kn}{2L+1}} \quad h[n] = \frac{1}{2L + 1} \sum_{k=-L}^{L} H_k e^{i \frac{2\pi kn}{2L+1}}.$$

define the discrete time- and bandlimiting operator as

$$\mathcal{D}h = \begin{cases} h[n] & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases} = Dh$$

$$\mathcal{B}h = \frac{1}{2L + 1} \sum_{k=-N}^{N} H_k e^{i \frac{2\pi kn}{2L+1}} = Bh.$$

Then the P-DPSSs are the eigenvectors of

$$\lambda_i \phi_{N,M,i} = BD \phi_{N,M,i}.$$
Properties

- $\lambda_i \phi_{N,M,i}$ and $D \phi_{N,M,i}$ are DFT pairs.
- $\sum_{k=-M}^{M} \phi_{N,M,i}[k]^2 = \lambda_i$
- Eigenvalues decay exponentially after $\sim (2N + 1)T$ eigenvalues.
- $\phi_{N,M,i}$ satisfies a second order difference equation.
- The matrix $DBD$ is equal to the normal matrix $A'A$ of the discrete Fourier Extension problem.
- The difference operator $T_n$ commutes with $A'A$, easy computation of $\phi_{M,N,i}$
- The left- and right singular vectors of $A$ are given by $\phi_{N,M,i}$ and $\phi_{M,N,i}$ respectively.
For any prolate type object, we have:

- prolate type object is eigenfunction of $BD$
- Time localisation proportional to exponential decaying $\lambda_i$
- Double orthogonality
- Frequency transform is another prolate type object
- $BD$ commutes with a second order differential operator
Fast Algorithms

Singular Values

\[ I_1 \quad I_{\text{mid}} \quad I_\epsilon \]

General Principle

- Isolate and solve for the middle part, at \( O(\log N) \) cost.
- Exploit the good condition of the first part.
- Truncate the last part.
Exploiting frequency properties

- Frequency localization of singular vectors (left & right), proportional to corresponding singular value

\[ A = USV^*, \quad C = \text{DCT-matrix} \]

\[
CU \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix}
\]

\[
CA \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix} \begin{bmatrix} S_1 V_1^* \\ S_2 V_2^* \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}
\]
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Introduction

PSWFS & FE

Cont. - Cont.
Cont. - Discr.
Discr. - Discr.

Fast Algorithms

Frequency

Commutation results

Extensions & Open Problems

Exploiting frequency properties

DCT result

\[
CA = \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}, \quad Cb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ where}
\]

\[
\cdot \kappa(\overline{A}_1) \approx 1, \text{ full rank}
\]

\[
\cdot \kappa(\overline{A}_2) \approx \epsilon^{-1}, \quad \text{rank}(\overline{A}_2) = O(\log N)
\]

Algorithm using random matrices

1. Solve \( \overline{A}_1 a_1 = \overline{b}_1 \) iteratively
2. Solve \( \overline{A}_1 c = \overline{A}_1 r \), for a number of random vectors \( r \)
   - \( r - c \) is in \( \text{null}(\overline{A}_1) \)
3. Construct orthogonal basis for \( \{\overline{A}_2 (r_i - c_i)\} \)
4. Solve \( \overline{A}_2 a_2 = \overline{b}_2 - \overline{A}_2 a_1 \) with \( a_2 \in \text{null}(\overline{A}_1) \)
5. \( a = a_1 + a_2 \)
Exploiting commuting operator

- Both left and right singular vectors of $A$ are P-DPSS
- Split eigenvalues of $A$.

- Splitted SVD, $\Sigma_1 \approx I$ and $\Sigma_\epsilon \approx 0$

$$A = \begin{bmatrix} U_1 & U_{\text{mid}} & U_\epsilon \end{bmatrix} \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_{\text{mid}} & \\ & & \Sigma_\epsilon \end{bmatrix} \begin{bmatrix} V_1 & V_{\text{mid}} & V_\epsilon \end{bmatrix}'$$

$$\approx U_1 V_1' + U_{\text{mid}} \Sigma_{\text{mid}} V_{\text{mid}}'$$
Exploiting commuting operator

Orthogonal solutions

Split solution $a$ into distinct parts $a_1 \in \text{span}\{V_1\}$, $a_{mid} \in \text{span}\{V_{mid}\}$, so that:

$$U_1 \Sigma_1 V'_1 a_1 = b_1, \quad U_{mid} \Sigma_{mid} V'_{mid} a_{mid} = b_{mid}.$$ 

The trick here is that for $b_1$, $a_1 = V_1 \Sigma_1^{-1} U'_1 b_1 = A'b_1$.

Algorithm

1. Find $U_{mid}, V_{mid}, \Sigma_{mid}$ using the tridiag. matrix
2. $a_{mid} = V_{mid} \Sigma_{mid}^{-1} U_{mid}^T b$
3. $b_1 = b - A_{mid} a_{mid}$
4. $a_1 = A'b_1$
5. $a = a_1 + a_{mid}$
• Accuracy overall is a bit worse than the symmetry-exploiting algorithm by M. Lyon
• Convergence seems to start slightly slower
- All algorithms are $O(N \log(N)^2)$
- At least for large $N$, symmetric and commutation are close
Smooth functions

Sobolev smoothing of the coefficients by solving

\[ Da \approx Db \quad \text{s.t.} \quad V_{\text{mid}}' a = 0. \]

using QR orthogonalization.
When extending a function on a rectangle, tensor-product structure can be exploited.

This algorithm has complexity $O(N \log N)$ for a total of $N$ points.
Easy 2D - Smooth solutions

The 1D smoothing can be applied to obtain a smooth extension

- Accurate and smooth solution for well-behaved functions
Easy 2D - Smooth solutions (bis)

The 1D smoothing can be applied to obtain a smooth extension.

- For more difficult functions, smoothness is limited to the borders.
Difficult 2D - General Domains

PSWFs generalise (at least partially) to any domain

- Is there symmetry to exploit?
- Does a commuting operator exist outside of tensor-product domains?
- What about frequency domain localisation?