Fast Fourier Extension with Discrete Prolate Spheroidal Sequences

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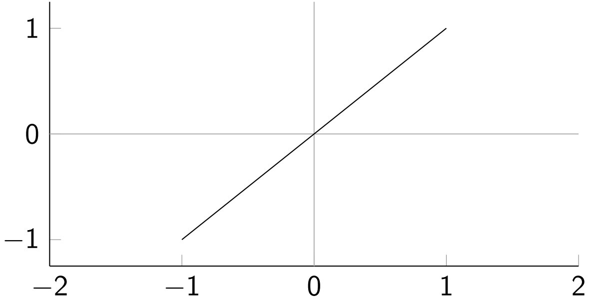
Fourier Extension

Fourier Extension principle

Constructing an exponentially convergent approximation of a function on a given interval, by fitting a Fourier series on a larger domain.

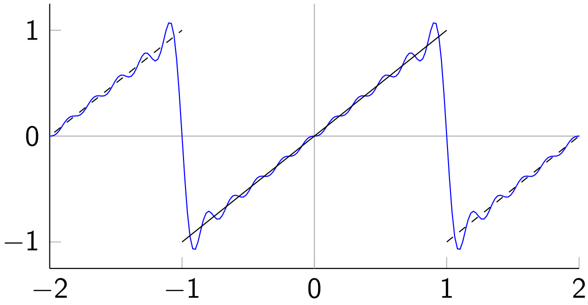
Example $f(x) = x$ on $[-1, 1]$



Fourier Extension

Fourier Series

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos 2\pi nx + b_n \sin 2\pi nx\right]$$



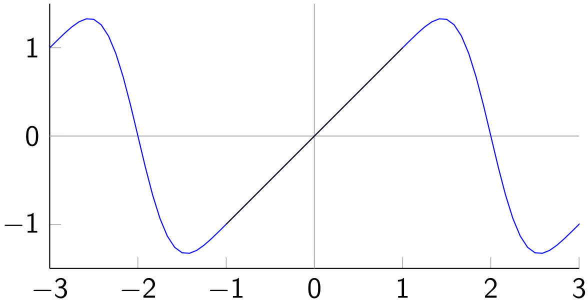
Fourier Extension

Fourier Extension

Fourier series over larger domain $[-T, T]$.

$$g \in G_N : g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos 2\pi \frac{nx}{T} + b_n \sin 2\pi \frac{nx}{T} \right]$$

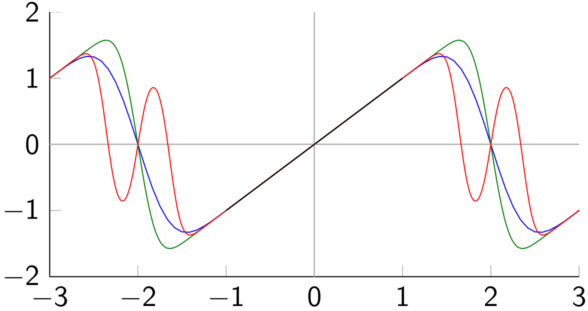
$$g_N := \arg \min_{g \in G_N} ||f - g||_{L^2_{[-1,1]}}$$



Fourier Extension

Redundancy

- Fourier basis restricted to interval constitutes a frame
- Least squares fitting problem has many good approximate solutions



Why Fourier Extensions

- Exponential convergence for smooth but non-periodic functions[1]
- Good resolution power (dof per wavelength) when T is close to 1[2]
- Approximation from uniformly spaced data
- A fast $O(N \log(N))$ algorithm exists[3], using FFTs, for the special case $T = 2$
  - This talk : A fast solver for general T

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[1] **Huybrechs 2010; Adcock**, **Huybrechs**, **Martin-Vaquero 2014**.

[2] **Adcock**, **Huybrechs 2011**.

[3] **Lyon 2012**.

Fourier Extension

Notation

- Frame basis functions $\phi_n(x) = \frac{1}{\sqrt{2T}} e^{i\frac{\pi n}{T}x}, n = -N, \ldots, N$
- Approximation $g(x) = \sum_{k=-N}^{N} a_k \phi_k(x)$

Fitting the Fourier Extension

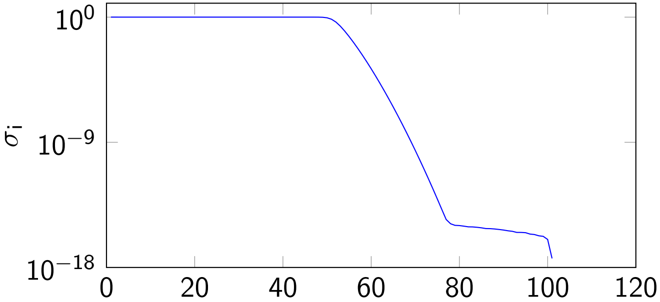
- Collocation at equidistant points $x_j = \frac{j}{M}, j = -M, \ldots, M$.
- Least squares problem $A_{ij} = \phi_i(x_j)$, $b_j = f(x_j)$

$$Aa \approx b$$

- Some oversampling $M = \eta N$ required, A is rectangular
- Goal: A fast solver for this system

A closer look at A

Singular values of A



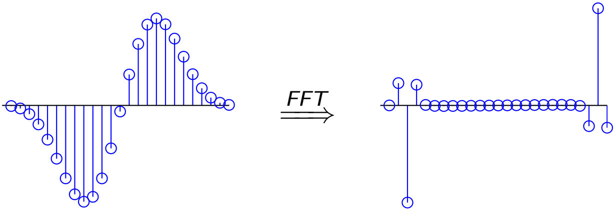
- Least-squares system $Aa = b$ is extremely ill-conditioned, $\kappa(A) = \mathcal{O}(\epsilon^{-1})$
- Solving with Truncated SVD or MATLAB backslash still gives approximation to machine accuracy[4]

[4] **Adcock**, **Huybrechs**, **Martin-Vaquero 2014**.

A closer look at A

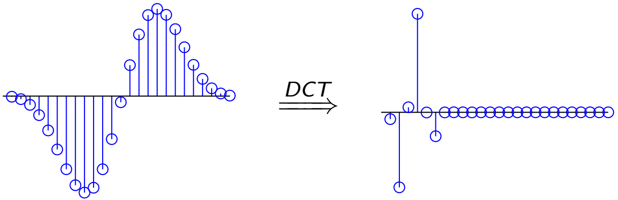
Singular vectors of A

- Are Periodic Discrete Prolate Spheroidal Sequences
- Corresponding singular values:
  - $O(N/T)$ are in $[1-\epsilon, 1)$
  - $O(N-N/T)$ are in $(0, \epsilon]$
  - $O(\log N)$ "intermediate" values in $(\epsilon, 1-\epsilon)$
- Easily obtained as eigenvectors of tridiagonal matrix
- Optimal frequency localisation w.r.t. FFT



Separating the singular values

- Frequency localization of singular vectors (left & right), energy in first N/T bins is proportional to σ_i



- $A = USV^*$, C=DCT-matrix

$$CU \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix}$$

$$CA \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix} \begin{bmatrix} S_1 V_1^* \\ S_2 V_2^* \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}$$

Separating the singular values

System CAa=Cb

- $CA = \begin{bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{bmatrix}$, $Cb = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \end{bmatrix}$, where
  - $\kappa(\overline{A}_1) \approx 1$, full rank
  - $\kappa(\overline{A}_2) \approx \epsilon^{-1}$, $rank(\overline{A}_2) = O(\log N)$

Algorithm using random matrices

1. Solve $\overline{A}_1 a_1 = \overline{b}_1$ iteratively
2. Solve $\overline{A}_2 a_2 = \overline{b}_2 - \overline{A}_2 a_1$, with $a_2 \in null(\overline{A}_1)$
   - Solve $\overline{A}_1 q = \overline{A}_1 r$, for $O(\log N)$ random vectors r
     - $r - q$ is in $null(\overline{A}_1)$
   - Construct reduced system $L = \overline{A}_2(R - Q)$
   - Solve $Lz = \overline{b}_2 - \overline{A}_2 a_1$ with an SVD
   - Expand z : $a_2 = (R - Q)z$
3. $a = a_1 + a_2$

Separating the singular values

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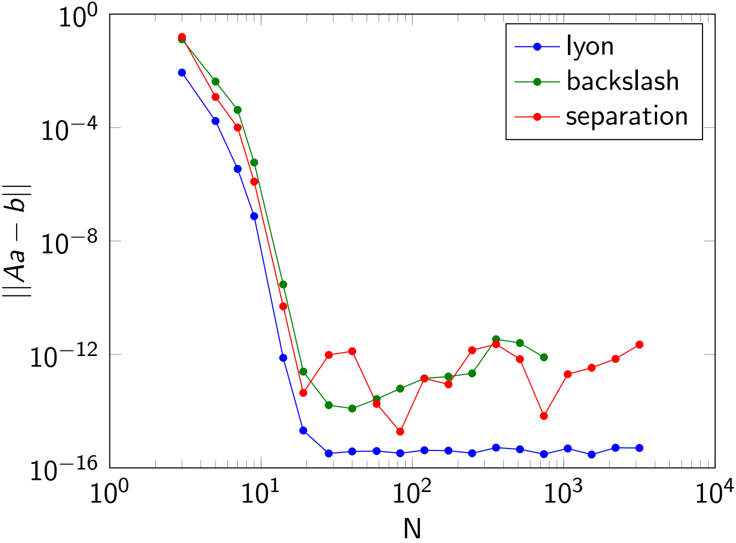
Algorithm for general T

Conclusions

Results



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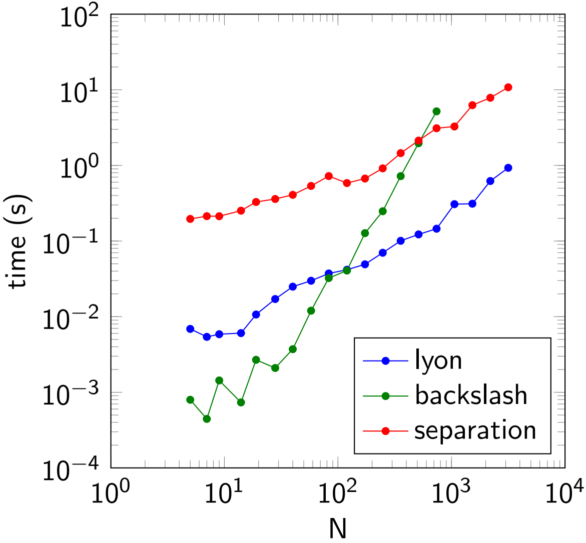
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
time (s)
10^0
10^1
10^2
10^3
10^4
N
lyon
backslash
separation

Conclusions

- Separation algorithm on par with M. Lyon algorithm on complexity and convergence
- Constant factor slower in practice, but more flexible
  - Possible to vary T with M
- P-DPSS provide theoretical framework for the FE method for equispaced data