Fast Fourier Extension with Discrete Prolate Spheroidal Sequences

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Fourier Extension

Fourier Extension principle

Constructing an exponentially convergent approximation of a function on a given interval, by fitting a Fourier series on a larger domain.

Example $f(x) = x$ on $[-1, 1]$
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Fourier Extension

A closer look at $A$

Algorithm for general $T$

Conclusions

Fourier Extension

Fourier Series

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos 2\pi nx + b_n \sin 2\pi nx]$$
Fourier Extension

Fourier Extension

Fourier series over larger domain $[-T, T]$.

$$g \in G_N : g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[ a_n \cos \frac{2\pi nx}{T} + b_n \sin \frac{2\pi nx}{T} \right]$$

$$g_N : = \arg \min_{g \in G_N} \| f - g \|_{L^2_{[-1,1]}}$$
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Redundancy

- Fourier basis restricted to interval constitutes a frame
- Least squares fitting problem has many good approximate solutions
Why Fourier Extensions

- Exponential convergence for smooth but non-periodic functions\(^1\)
- Good resolution power (dof per wavelength) when \(T\) is close to 1\(^2\)
- Approximation from uniformly spaced data
- A fast \(O(N \log(N))\) algorithm exists\(^3\), using FFTs, for the special case \(T = 2\)
  - This talk: A fast solver for general \(T\)

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\(^1\) Huybrechs 2010; Adcock, Huybrechs, Martin-Vaquero 2014.
\(^2\) Adcock, Huybrechs 2011.
\(^3\) Lyon 2012.
Notation

- Frame basis functions $\phi_n(x) = \frac{1}{\sqrt{2T}} e^{i\frac{\pi n x}{T}}, n = -N, \ldots, N$
- Approximation $g(x) = \sum_{k=-N}^{N} a_k \phi_k(x)$

Fitting the Fourier Extension

- Collocation at equidistant points $x_j = \frac{j}{M}, j = -M, \ldots, M$.
- Least squares problem $A_{ij} = \phi_i(x_j), b_j = f(x_j)$

$$Aa \approx b$$

- Some oversampling $M = \eta N$ required, $A$ is rectangular
- Goal: A fast solver for this system
A closer look at \( A \)

Singular values of \( A \)

- Least-squares system \( Aa = b \) is extremely ill-conditioned, \( \kappa(A) = \mathcal{O}(\epsilon^{-1}) \)
- Solving with Truncated SVD or MATLAB backslash still gives approximation to machine accuracy\(^4\)

\(^4\)Adcock, Huybrechs, Martin-Vaquero 2014.
A closer look at $A$

Singular vectors of $A$

- Are Periodic Discrete Prolate Spheroidal Sequences
- Corresponding singular values:
  - $O(N/T)$ are in $[1 - \epsilon, 1)$
  - $O(N - N/T)$ are in $(0, \epsilon]$
  - $O(\log N)$ “intermediate” values in $(\epsilon, 1 - \epsilon)$
- Easily obtained as eigenvectors of tridiagonal matrix
- Optimal frequency localisation w.r.t. FFT
Separating the singular values

- Frequency localization of singular vectors (left & right), energy in first $N/T$ bins is proportional to $\sigma_i$.

- $A = USV^*$, $C =$ DCT-matrix

\[
CU \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix} \\
CA \approx \begin{bmatrix} F_1 & \epsilon \\ \epsilon & F_2 \end{bmatrix} \begin{bmatrix} S_1 V_1^* \\ S_2 V_2^* \end{bmatrix} = \begin{bmatrix} F_1 S_1 V_1^* + O(\epsilon) \\ F_1 S_2 V_2^* + O(\epsilon) \end{bmatrix}
\]
Separating the singular values

System $CAa = Cb$

- $CA = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix}$, $Cb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, where
  - $\kappa(\bar{A}_1) \approx 1$, full rank
  - $\kappa(\bar{A}_2) \approx \epsilon^{-1}$, $\text{rank}(\bar{A}_2) = O(\log N)$

Algorithm using random matrices

1. Solve $\bar{A}_1 a_1 = b_1$ iteratively
2. Solve $\bar{A}_2 a_2 = b_2 - \bar{A}_2 a_1$, with $a_2 \in \text{null}(\bar{A}_1)$
   - Solve $\bar{A}_1 q = \bar{A}_1 r$, for $O(\log N)$ random vectors $r$
     - $r - q$ is in $\text{null}(\bar{A}_1)$
   - Construct reduced system $L = \bar{A}_2 (R - Q)$
   - Solve $Lz = b_2 - \bar{A}_2 a_1$ with an SVD
   - Expand $z : a_2 = (R - Q)z$
3. $a = a_1 + a_2$
Separating the singular values

System \( CAa = Cb \)

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  - \( \kappa(\bar{A}_2) \approx \epsilon^{-1} \), \( \text{rank}(\bar{A}_2) = O(\log N) \)

Algorithm using random matrices

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   - Solve \( \bar{A}_1 q = \bar{A}_1 r \), for \( O(\log N) \) random vectors \( r \)
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Results

\[ \| Aa - b \| \]

- Lyon
- Backslash
- Separation

\[ N \]

\[ 10^{-16} \]

\[ 10^{-12} \]

\[ 10^{-8} \]

\[ 10^{-4} \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]

\[ 10^4 \]
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Results

\[ \text{time (s)} \]

\[ N \]

\[ \text{lyon} \]

\[ \text{backslash} \]

\[ \text{separation} \]
Conclusions

- Separation algorithm on par with M. Lyon algorithm on complexity and convergence
- Constant factor slower in practice, but more flexible
  - Possible to vary $T$ with $M$
- P-DPSS provide theoretical framework for the FE method for equispaced data