Korobovs algorithms for Lattice rules

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Introduction

N. M. Korobov (1917-2004)

- 1982 paper *"On the Computation of Optimal Coefficients"*
- 3 Constructions of rank-1 lattice rules with optimal convergence in a Korobov space
- Fast algorithms, complexity $O(sN\log(N))$

Overview

Introduction

Rank-1 Lattice rules

- Generator $\mathbf{a} \in \mathbb{Z}_N^s$

$$P_N = \left\{ \left\{ \frac{k\mathbf{a}}{N} \right\} : k = 0, 1, \ldots, N-1 \right\}$$

$$Q_N(f) = \frac{1}{N} \sum_{x_k \in P_N} f(x_k)$$

- Desirable properties
  - Optimal error convergence $O(N^{-1} \log^{\beta(s)}(N))$
  - Fast construction
  - Extensibility in N and s

Constructions

Representation

- Base 2 representation, $a_1 = (a_{1v} \cdots a_{12} a_{11})_2$

$$\begin{matrix} a_{1n} & \cdots & a_{1v} & \cdots & a_{12} & a_{11} \\ a_{2n} & \cdots & a_{2v} & \cdots & a_{22} & a_{21} \\ \vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\ a_{rn} & \cdots & a_{rv} & \cdots & a_{r2} & a_{r1} \\ \vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\ a_{sn} & \cdots & a_{sv} & \cdots & a_{s2} & a_{s1} \end{matrix}$$

- Indices $1 \leq r \leq s$ and $1 \leq v \leq n$, $N = 2^n$

Recent Constructions

CBC

- Generator **a** constructed component by component

$$\begin{matrix} a_{1n} & \cdots & a_{1v} & \cdots & a_{12} & a_{11} \\ a_{2n} & \cdots & a_{2v} & \cdots & a_{22} & a_{21} \\ \vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\ a_{rn} & \cdots & a_{rv} & \cdots & a_{r2} & a_{r1} \\ \vdots & \iddots & \vdots & \iddots & \vdots & \vdots \end{matrix} \quad \downarrow r=1,\dots,s$$

- Korobov 1959, Sloan en Reztsov 2002
- Extensible in dimension s
- Fast CBC (Nuyens) $O(sN\log(N))$

Recent Constructions

Neiderreiter and Pillichshammer (2009)

- Generator **a** constructed digit by digit

$$\overset{v=2,\ldots,n}{\longleftarrow}$$

$$\begin{array}{ccccc}
\cdots & a_{1v} & \cdots & a_{12} & a_{11} \\
\cdots & a_{2v} & \cdots & a_{22} & a_{21} \\
\iddots & \vdots & \iddots & \vdots & \vdots \\
\cdots & a_{rv} & \cdots & a_{r2} & a_{r1} \\
\iddots & \vdots & \iddots & \vdots & \vdots \\
\cdots & a_{sv} & \cdots & a_{s2} & a_{s1}
\end{array}$$

- Extensible in $N = 2^n$
- No optimal convergence proven for base 2
- $O(s2^s N)$

Korobov Constructions

DBD

- Generator **a** constructed digit by digit

$$
\overset{v=2,\ldots,n}{\longleftarrow}
$$

$$
\begin{matrix}
a_{1n} & \cdots & a_{1v} & \cdots & a_{12} & a_{11} \\
a_{2n} & \cdots & a_{2v} & \cdots & a_{22} & a_{21} \\
\vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\
a_{rn} & \cdots & a_{rv} & \cdots & a_{r2} & a_{r1} \\
\vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\
a_{sn} & \cdots & a_{sv} & \cdots & a_{s2} & a_{s1}
\end{matrix}
$$

- Niederreiter and Pillichshammer principle, but no extensibility in N
- $O(s2^sN)$

Korobov Constructions

DBD+CBC

- Generator **a** constructed digit by digit, digits are added component by component

$$\overleftarrow{\qquad v=2,\dots,n \qquad}$$

$$\begin{array}{cccccc}
a_{1n} & \cdots & a_{1v} & \cdots & a_{12} & a_{11} \\
a_{2n} & \cdots & a_{2v} & \cdots & a_{22} & a_{21} \\
\vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\
a_{rn} & \cdots & a_{rv} & \cdots & a_{r2} & a_{r1} \\
\vdots & \iddots & & \iddots & \vdots & \vdots \\
a_{sn} & & & \cdots & a_{s2} & a_{s1}
\end{array}
\quad \Big\downarrow\, r=1,\dots,s$$

- Faster version of DBD
- $O(sN)$ complexity, with $O(sN)$ memory requirement

Korobov Constructions

CBC+DBD

- Generator **a** constructed component by component, components are added digit by digit

$$
\xleftarrow{\;v=2,\ldots,n\;}
$$

$$
\begin{array}{cccccc}
a_{1n} & \cdots & a_{1v} & \cdots & a_{12} & a_{11} \\
a_{2n} & \cdots & a_{2v} & \cdots & a_{22} & a_{21} \\
\vdots & \iddots & \vdots & \iddots & \vdots & \vdots \\
 & & a_{rv} & \cdots & a_{r2} & a_{r1} \\
 & \iddots & \vdots & \iddots & \vdots & \vdots
\end{array}
\quad \Big\downarrow r=1,\ldots,s
$$

- Extensible in dimension s
- $O(sN\log(N))$

Quality criterion

RKHS

- worst-case error

$$e(Q_N, \mathcal{F}) := \sup_{\substack{f \in \mathcal{F} \\ ||f||_{\mathcal{F}} \leq 1}} |I(f) - Q_N(f)|$$

- Algorithms are tailored to function space

Classical theory - Korobov space

- E_α^s, the space of $[0,1)^s$-periodic functions f for which

$$|\hat{f}(\mathbf{h})| \leq c r(\mathbf{h})^{-\alpha} \qquad r(\mathbf{h}) = \prod_{i=1}^{s} \max(1, |h_i|)$$

$$P_\alpha(\mathbf{a}, n) = \sum_{\substack{\mathbf{h} \in \mathbb{Z}^s \setminus \{\mathbf{0}\} \\ \mathbf{h} \cdot \mathbf{a} \equiv 0 \pmod{n}}} \frac{1}{r(\mathbf{h})^\alpha}.$$

Quality criterion

Classical theory - ctd.

- Dual lattice

$$L^{\perp} := \{\mathbf{m} \in \mathbb{R}^s : \mathbf{m} \cdot \mathbf{x} \in \mathbb{Z} \text{ for all } \mathbf{x} \in L\}.$$

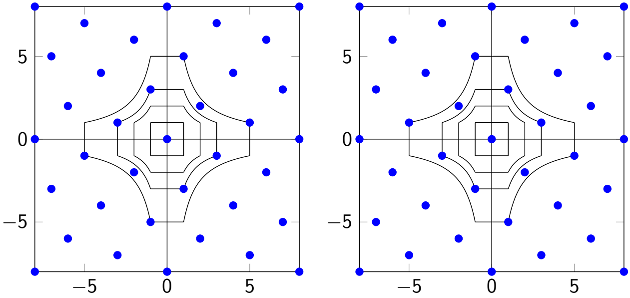
- Quality criteria

$$P_\alpha(\mathbf{a}, n) = \sum_{\mathbf{0} \neq \mathbf{h} \in L^{\perp}} \frac{1}{r(\mathbf{h})^\alpha}$$

$$R(\mathbf{a}, n) = \sum_{\mathbf{0} \neq \mathbf{h} \in L^{\perp} \cap (-N/2, N/2]^s} \frac{1}{r(\mathbf{h})}$$

$$q(L) = \min_{\mathbf{0} \neq \mathbf{h} \in L^{\perp}} r(\mathbf{h})$$

Proofs



(a) $L^\perp \to q$
difficult

(b) $N \cdot L \to Q$
easy

Gel'fond Lemma

The inequalities $Q \geq C_1 q^s$ and $q \geq C_1 \frac{Q^s}{N^{s^2-1}}$ hold, with a positive constant $C_1 = C_1(s)$.

Proofs

Korobov Theorem

$1, a_1, \ldots, a_s$ are optimal coefficients for N points iff for $k \neq 0$ and $-\lfloor (N-1)/2 \rfloor \leq k \leq \lfloor N/2 \rfloor$

$$|k| \left\| \frac{a_1 k}{N} \right\| \cdots \left\| \frac{a_s k}{N} \right\| \geq \frac{Q}{N^s} \geq \frac{1}{(B_1 \ln^{\beta(s)} N)}$$

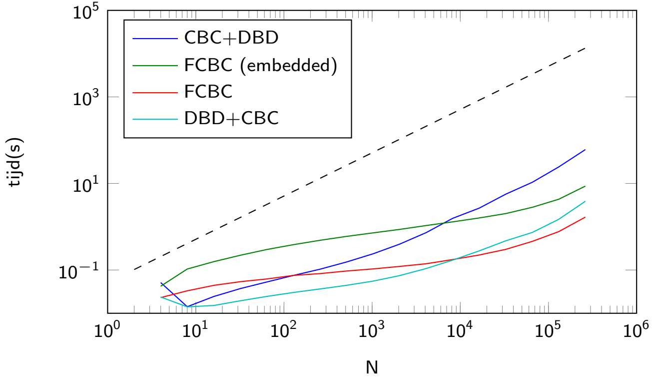
Objective function

$$h_v(\mathbf{a}) = \frac{1}{2^v} \sum_{\substack{m=1 \\ m \equiv 1 \pmod 2}}^{2^v} \prod_{j=1}^{s} \left(2n - 2v + \frac{1}{||ma_j/2^v||} \right)$$

Theorem bound is proven through an averaging argument.

Complexity

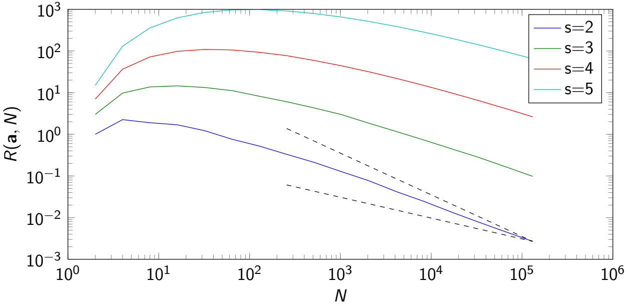
Complexity as a function of N, $s = 100$



- Fast CBC configured to minimise $R(\mathbf{a}, N)$, like the Korobov algorithms.

Quality

R-criterion



- Korobov proves $O(N^{-1}\log^{\beta(s)}(N))$ convergence, with dimension-dependent constant
- Korobov algorithms perform only marginally worse than FastCBC for $R(\mathbf{a}, N)$.

Introducing weights

Avoiding dimension-dependent constant in a weighted function space.

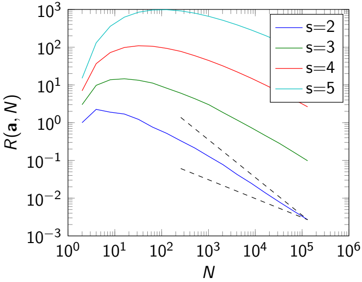
- Product weight kernel leads to weighted $R(\mathbf{a}, N)$ criterion

$$K(\mathbf{x}) = \prod_{j=1}^{s}(1 + \gamma_j \omega(x_j)),$$

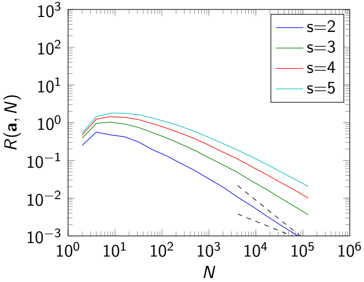
- Altered weight function

$$h_v(\mathbf{a}) = \frac{1}{2^v} \sum_{\substack{m=1 \\ m\equiv 1 \pmod 2}}^{2^v} \prod_{j=1}^{s} \left(2n - 2v + \gamma_j \frac{1}{||ma_j/2^v||}\right)$$

Introducing weights



(c) No weights



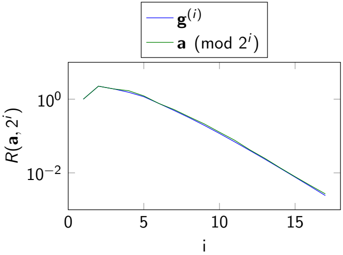
(d) Weights

- Theoretical justification is work in progress

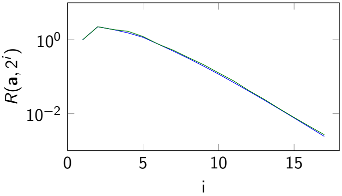
Embeddedness

- Generator is good for range of points $p^{m_1}, \ldots, p^{m_2}$
- First i digits are optimal generator for 2^i points
- Test by comparing with Fast CBC generator calculated for 2^i points

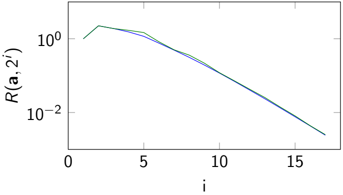
Embeddedness



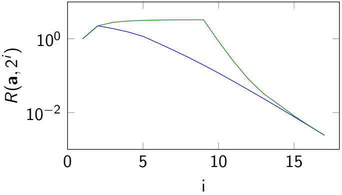
(e) DBD+CBC



(f) CBC+DBD



(g) FCBC (embedded)



(h) FCBC

Conclusions

- Algorithms show interesting combinations of component-by-component and digit-by-digit approaches
- Proofs are based on the Gel'fond lemma relating q and Q.
- Usability is limited to $R(\mathbf{a}, N)$ criterion, but extensions to weighted function spaces are subject of further work.

References

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