Fast Algorithms for Fourier Extensions

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Setting

Given a function f as N data points on an equispaced grid, find an expansion in Fourier basis functions

- converges fast (geometric if f is analytic)
- is easily computable, $O(N)$ computations.

Classical Fourier interpolation relies on periodicity for fast convergence.

Fourier interpolation for non-periodic functions exhibits the Gibbs phenomenon and converges very slowly.

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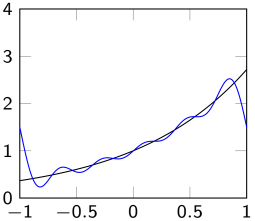
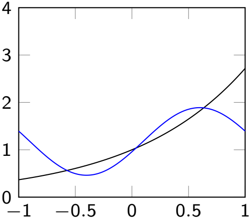


Fourier interpolation for non-periodic functions exhibits the Gibbs phenomenon and converges very slowly.

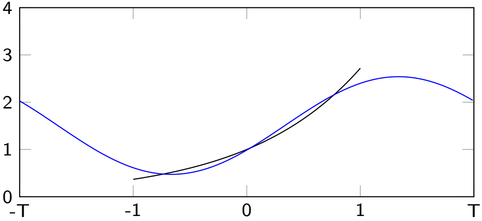
Fourier Extension

A Fourier Extension is an approximation of some function on a given domain in a Fourier Basis on a larger domain (a Fourier Frame).

Example: approximate $\exp(x)$ through interpolation and Fourier extension for increasing dof:



(a) Fourier interpolation



(b) Fourier extension

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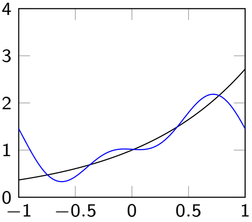
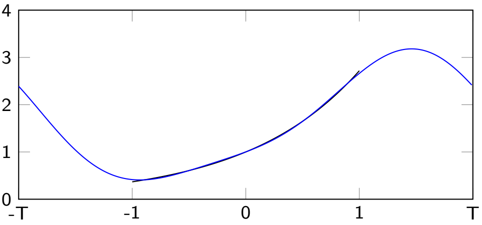


(c) Fourier interpolation



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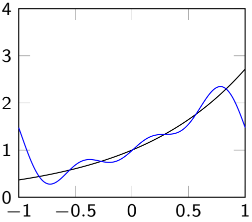
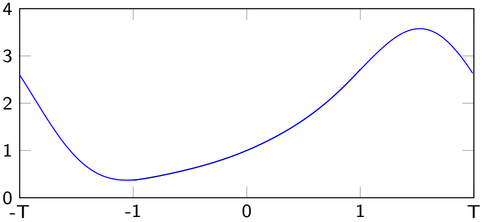


(e) Fourier interpolation



(f) Fourier extension

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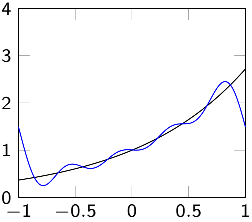
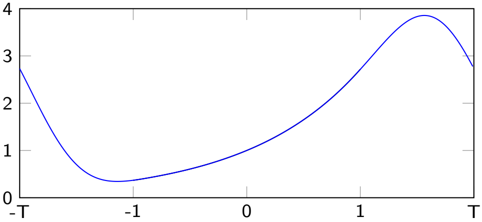


(g) Fourier interpolation



(h) Fourier extension

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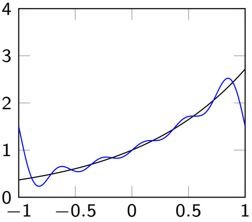
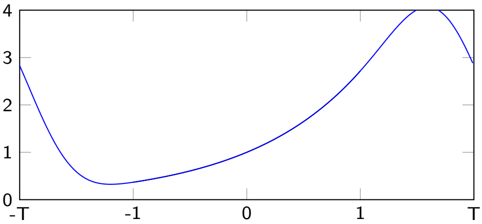


(i) Fourier interpolation



(j) Fourier extension

Fourier Extension

Notation

- Frame basis functions $\phi_m(x) = \frac{1}{\sqrt{2T}} e^{i\frac{\pi m}{T}x}, m = -M, \ldots, M$
- Approximation $g(x) = \sum_{k=-M}^{M} a_k \phi_k(x)$

Fitting the Fourier Extension

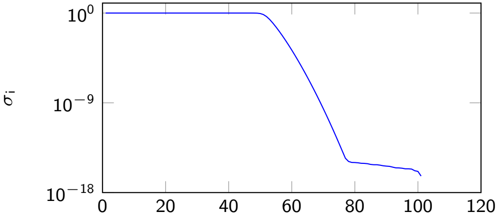
- Collocation at equidistant points $x_j = \frac{j}{N}, j = -N, \ldots, N$.
- Least squares problem $A_{ij} = \phi_i(x_j)$, $b_j = f(x_j)$

$$Aa = b$$

- Some oversampling $N = \eta M$ required, A is rectangular
- A is DFT-subblock
- Fast solver for $Aa = b$ needed
- Fast algorithm exists for $T = 2$ (Lyon 2011)

A closer look at A

Singular values of A



- Spectrum has a plateau shape (Slepian 1978, Wilson 1987)
- Size of transition/plunge/problematic region

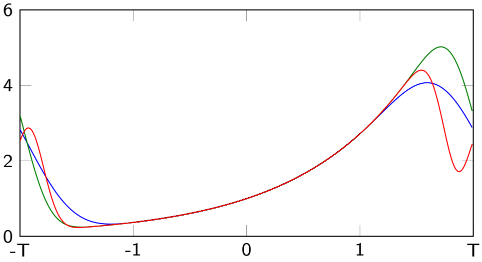
$$1 - \epsilon > \theta_i > \epsilon$$

  grows as $O(\log N)$.
- System matrix is extremely ill-conditioned

A closer look at A

Ill-conditioning explained

- *A Frame is a set of vectors that spans a vector space, but is not necessarily linearly independent.*
- Fourier basis restricted to interval constitutes a frame, a redundant basis
- Finding a representation is a very ill-conditioned problem

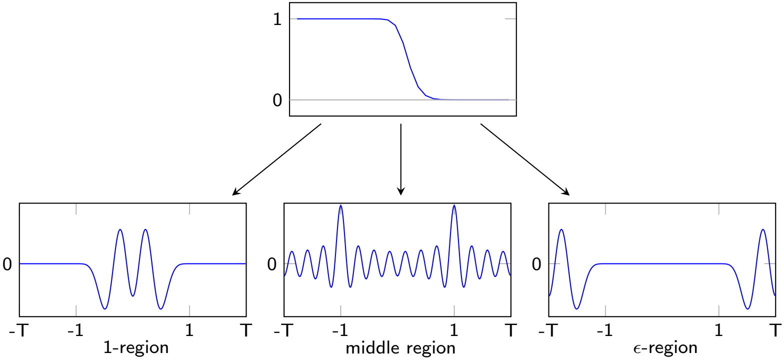


However, $||g - f||_{[-1,1]}$, $||h - f||_{[-1,1]}$, $||l - f||_{[-1,1]}$ are all small.

A closer look at A

Ill-conditioning explained

Fourier series corresponding to the singular vectors of A:



- Solving with Truncated SVD yields approximation to machine accuracy (Adcock, Huybrechs, Martin-Vaquero 2014)

Solving the least-squares system

Truncated SVD

- $A = U\Sigma V'$
- $x = V\Sigma^{-1}U'b$
- $x = \underbrace{V_1\Sigma_1^{-1}U_1'b_1}_{x_1} + \underbrace{V_{mid}\Sigma_{mid}^{-1}U_{mid}'b_{mid}}_{x_{mid}} + \underbrace{V_\epsilon\Sigma_\epsilon^{-1}U_\epsilon'b_\epsilon}_{x_\epsilon}$

Assumption: b_ϵ is negligible (discrete Picard condition)

x_1 is easy when x_{mid} is known

- $b_1 = b - b_{mid} = b - Ax_{mid}$
- $x_1 = V_1\Sigma_1^{-1}U_1'b_1 \approx V_1\Sigma_1 U_1'b_1 = A'b_1$

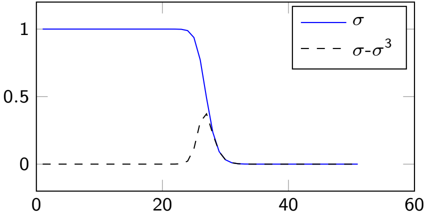
All you need is one application of A and A' (fast!)

How to find x_{mid}?

Isolating the problematic singular values

- 'Projection operator' $I - AA'$ isolates the middle singular values

$$(I - AA')A = U(\Sigma - \Sigma^3)V'$$



The numerical nullspace of $(I - AA')A$ contains both the 1 and ϵ-regions.

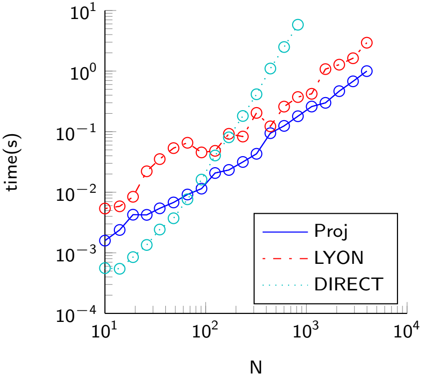
- Solving $(I - AA')Ax = (I - AA')b$ yields x_{mid}
- $(I - AA')A$ has numerical rank $O(\log N)$
- This can be solved efficiently in $O(N \log^2 N)$

Implementation

Algorithm

$$(I - AA')Ax_\beta = (I - AA')b$$
$$x_\alpha = A'(b - Ax_\beta)$$
$$x = x_\alpha + x_\beta$$

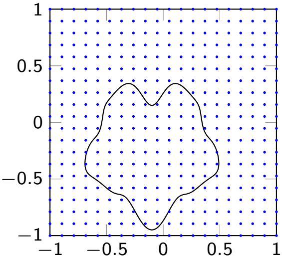
Results



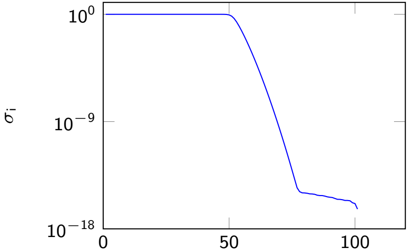
2D Frames

Fourier Frame

- A consists of selected rows of the 2D DFT matrix.



(k) Masked Grid

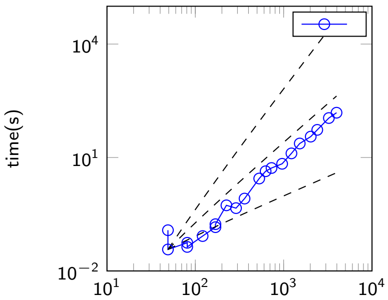


(l) Singular values of A

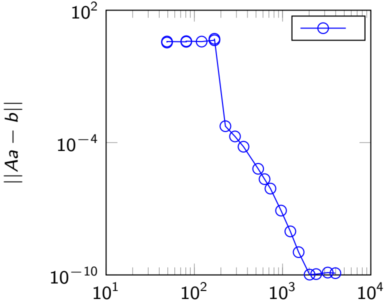
- Conjecture: the problematic region grows as $O(\sqrt{N})$ for N total points.
- The algorithm complexity becomes $O(N^2)$

2D Frames

Experimental results for 2D extensions:



(m) Timings

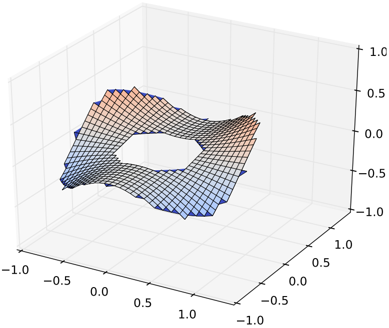


(n) Error

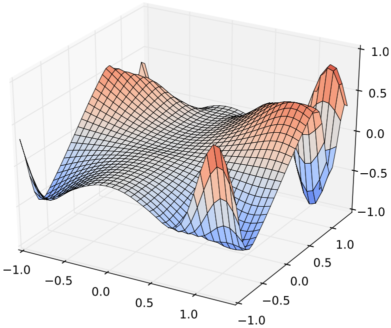
- Complexity is $O(N^2)$, as expected
- Convergence speed is seemingly conserved

2D Fourier Frame

$$f(x,y) = \frac{5xy^2}{x^2+y^2+4}, \quad D(x,y) = 1 > x^2 + y^2 > 0.4$$



(o) Frame Approximation

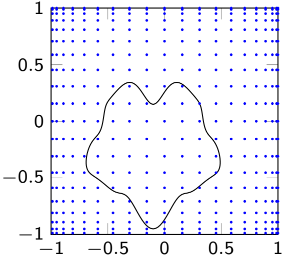


(p) Extension

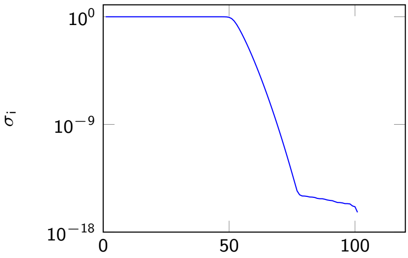
2D Frames

Chebyshev Frame

- A consists of selected rows of the 2D Chebyshev interpolation matrix.



(q) Masked Grid



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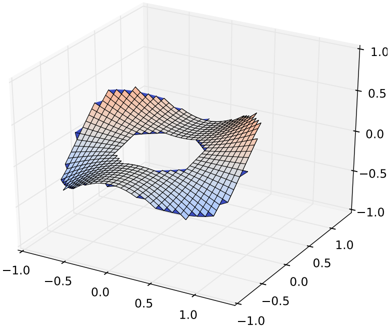
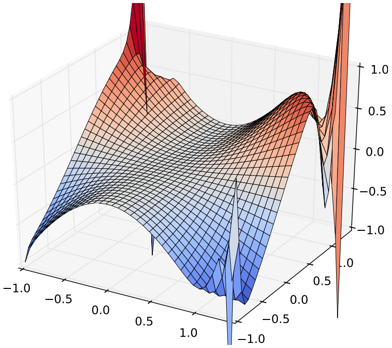


(s) Frame Approximation



(t) Extension

Conclusions

- Fourier extensions provide efficient and fast converging representations in a Fourier basis
- Our algorithm generalizes previous algorithms, is concise and competitive in speed.
- The algorithm generalizes to any (frame collocation) system that shows the plateau in the spectrum.
- 2D Frame extensions are promising, but still require $O(N^2)$ operations.