Fast approximation using Fourier Extensions

Roel Matthysen & Daan Huybrechs

KU Leuven

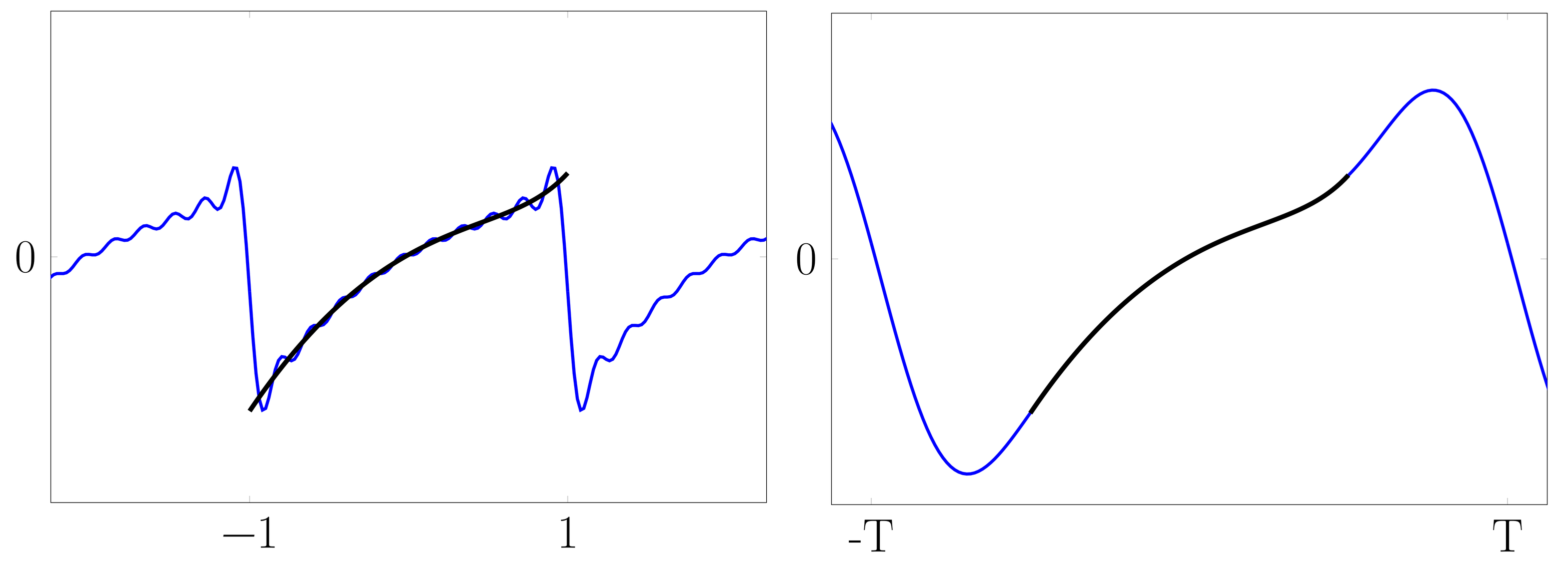
Fourier Extensions

Goal

Approximate a function $f(x)$ on $[-1, 1]$, with a Fourier series

$$f(x) \approx \sum_k a_k \phi_k(x), \quad \phi_k(x) = e^{ikcx}$$

- The traditional Fourier basis on $[-1, 1]$ suffers from the **Gibbs-phenomenon** and **slow convergence**.
- On an extended interval $[-T, T]$, the Fourier series can **converge exponentially** [1].



Left: function approximated by a classical Fourier series on $[-1, 1]$*. Right: function on* $[-1, 1]$ *approximated by a Fourier extension on* $[-T, T]$*.*

- Benefits of the method:
  - **Fast convergence** with the convenience of Fourier series.
  - **Good resolution power** for representing oscillatory functions.
  - **Equispaced data points** to avoid severe time-step restrictions.

Fast Algorithms & Ill-conditioning

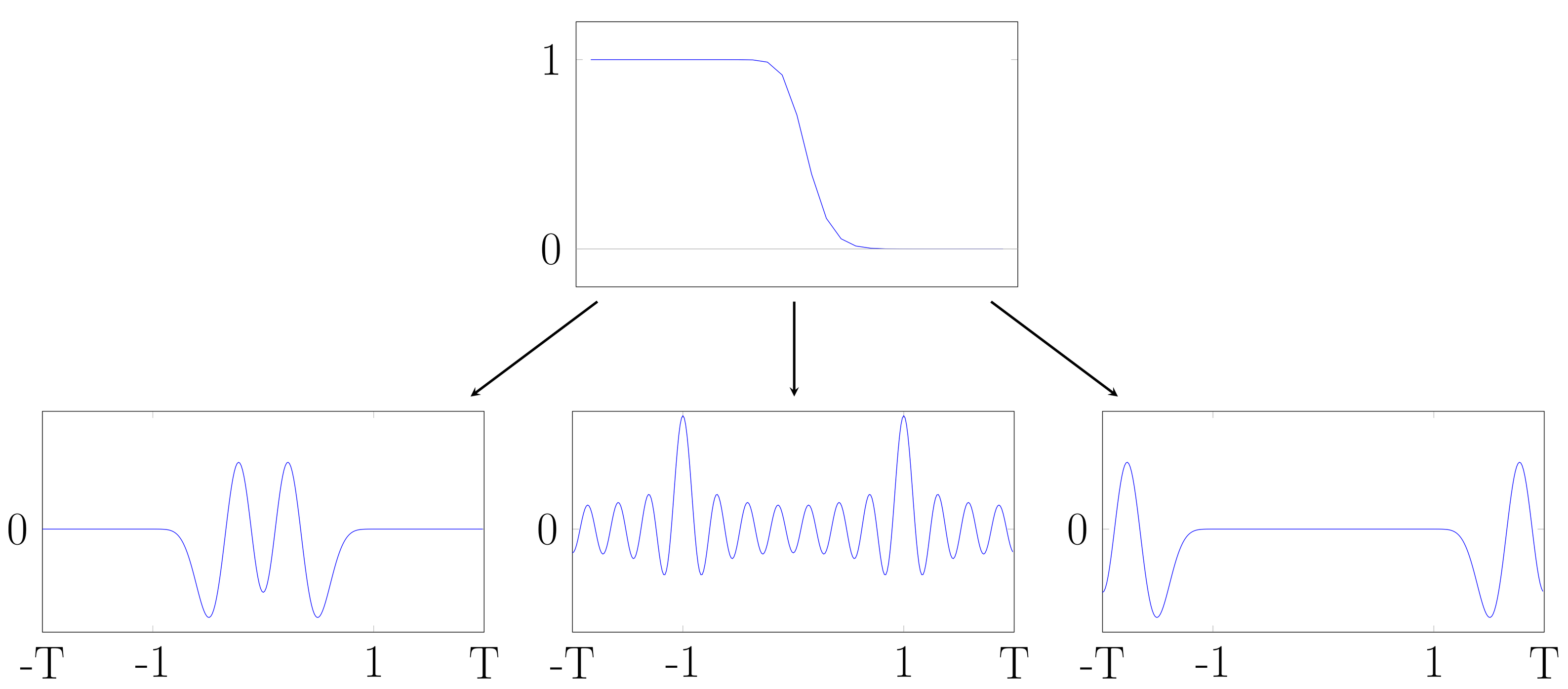
Goals

- Stable convergence up to machine precision.
- Match the $O(N \log N)$ complexity of the FFT.

- The Fourier coefficients are found by **collocation** on a uniform point set, leading to a least squares system

$$\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_M) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \ddots & \\ \phi_1(x_M) & & \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \qquad x_i \in [-1, 1].$$

- Problem: This system is **severely ill-conditioned**. This is clear from the **singular values** and the associated Fourier series:



Small singular values illustrate the **redundancy** in Fourier Extensions.

- In the limit for $N \to \infty$, the restricted Fourier basis constitutes a **frame**.

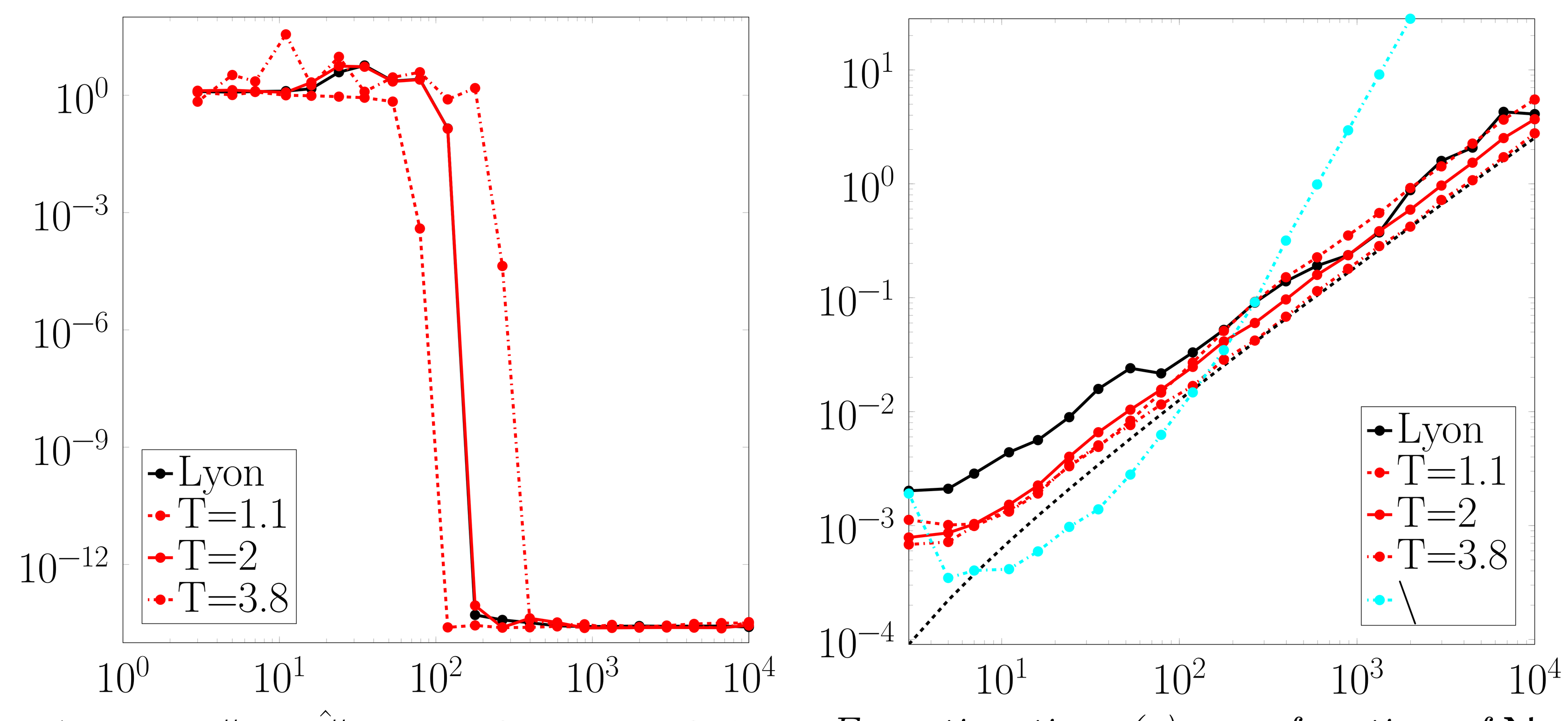
Theoretical connection

The SVD is related to discretised analogues of **Prolate Spheroidal Wave** theory [2]

- The plunge region grows as $O(\log N)$. Isolating the plunge region yields a **small problem**, and a **well-conditioned problem**. Both can be solved in $O(N \log^2 N)$ time.
- Isolation techniques follow from the theory:
  - Exploit **symmetries** present when $T = 2$ (M. Lyon)
  - Singular vectors obey a 2nd order **difference equation**. Calculate plunge region vectors directly.
  - Singular vectors have **compact frequency support**, allowing a split between the well-conditioned part and low-rank ill-conditioned part.

Numerical results

- Fourier Extension applied to an oscillatory function $f(x) = \mathrm{Ai}(34 - 70\cos x)$:



Accuracy $\|f - \hat{f}\|_\infty$ *as a function of* N. *Execution time (s) as a function of* N.

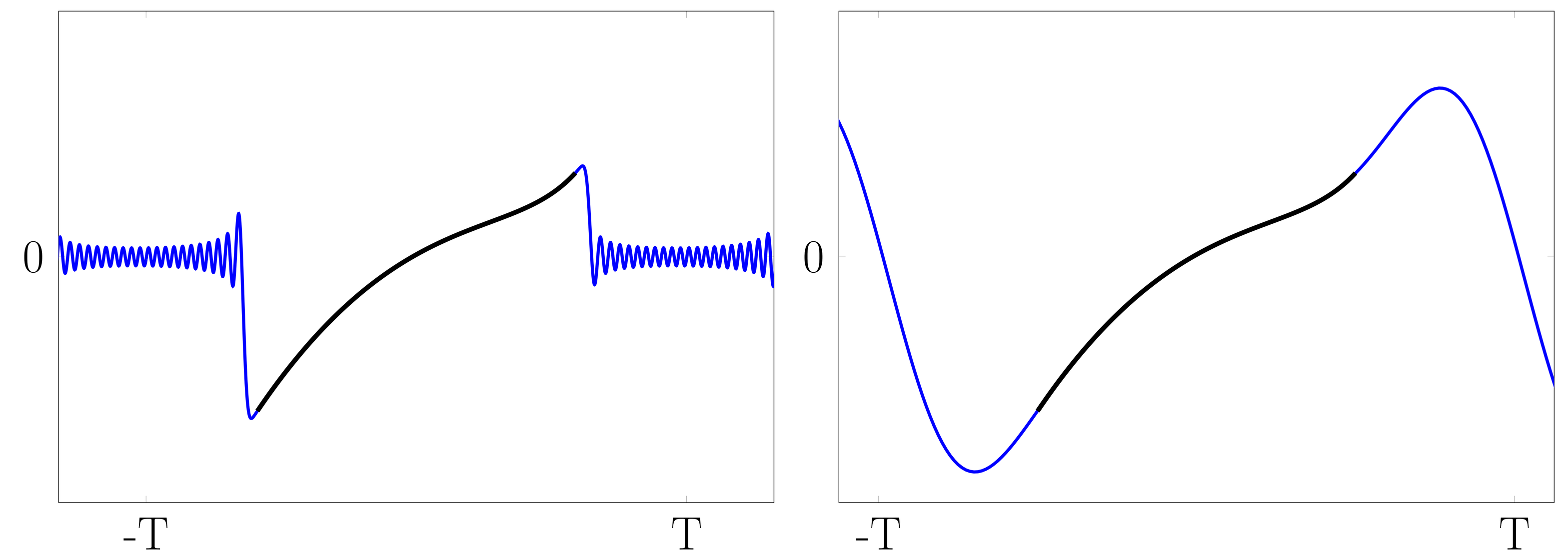
- Exponential convergence once the required N is passed, $O(N \log N)$ convergence confirmed.
- Increasing T increases rate of convergence, but convergence sets in later.
- To maintain stability, **oversampling** has to scale as $1/T$.

Smooth Extensions - w. B. Adcock, M. Lyon

Goal

Find extension $\hat{f}$ that minimises some (weighted) Sobolev norm $\|f - \hat{f}\|_{H^k}$.

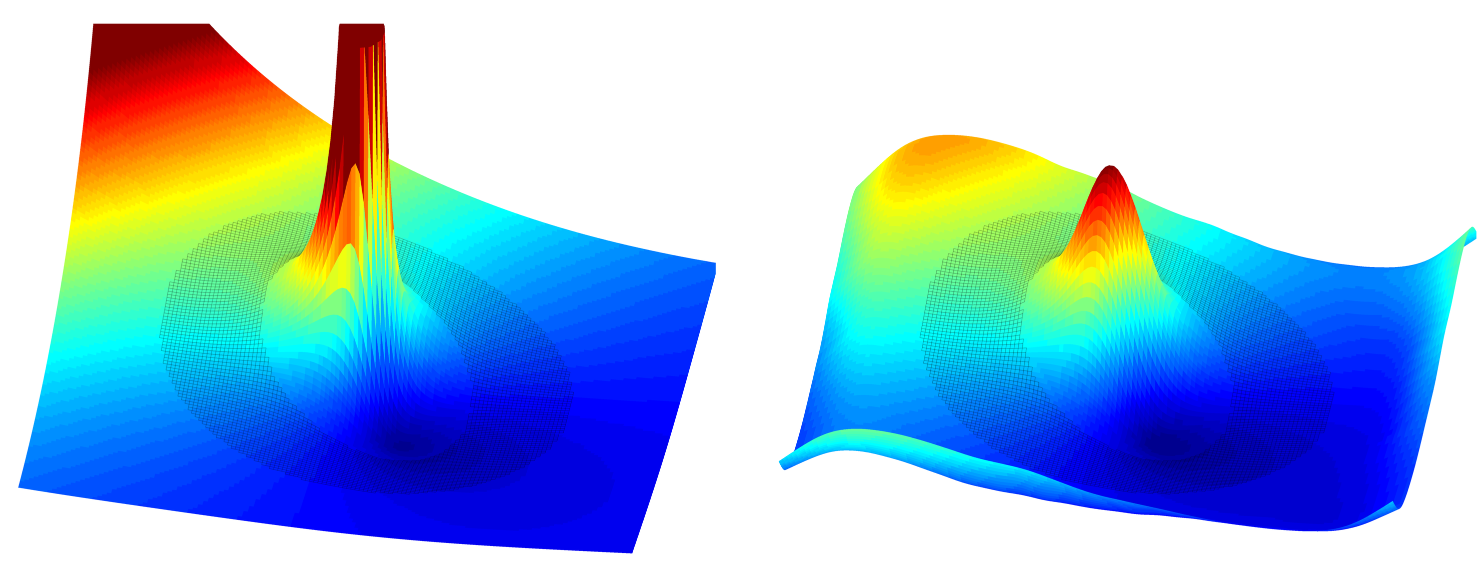
- Problem: Solution methods prefer small norm solutions, **extension converges to zero** outside interval.



- Ongoing research: **Exploit the redundancy** to smoothen out the solution.

2D Extensions

- Works for any set of collocation points, such as a **ring-shaped domain**. Function values outside the domain are never used.



- The plunge region grows as $O(\sqrt{N})$, limiting complexity to $O(N^2)$.

References

[1] Daan Huybrechs. On the Fourier extension of nonperiodic functions. *SIAM Journal on Numerical Analysis*, 47(6):4326–4355, 2010.

[2] D Slepian. Prolate spheroidal wave functions, Fourier analysis, and uncertainty -V: The Discrete Case. *Bell Syst. Tech. J*, 1978.

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