Frames and Numerical Approximation

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Frames

Definition: In a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, a set $\Phi = \{\phi_k : k \in I\} \subset \mathcal{H}$ is a frame if there exists *frame constants* $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k\in I} |\langle f, \phi_k\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}.$$

Equivalently: the *pre-frame operator*,

$$\mathcal{T} : \mathbf{c} \mapsto \sum_{k\in I} c_k \phi_k$$

is *bounded and onto* as a linear operator from $\ell^2(I)$ to $\mathcal{H}$.

Benefits: A frame generalises an orthonormal basis, but the set Φ does not need to be linearly independent. This gives us *more freedom* in defining a frame for the purposes of numerical approximation.

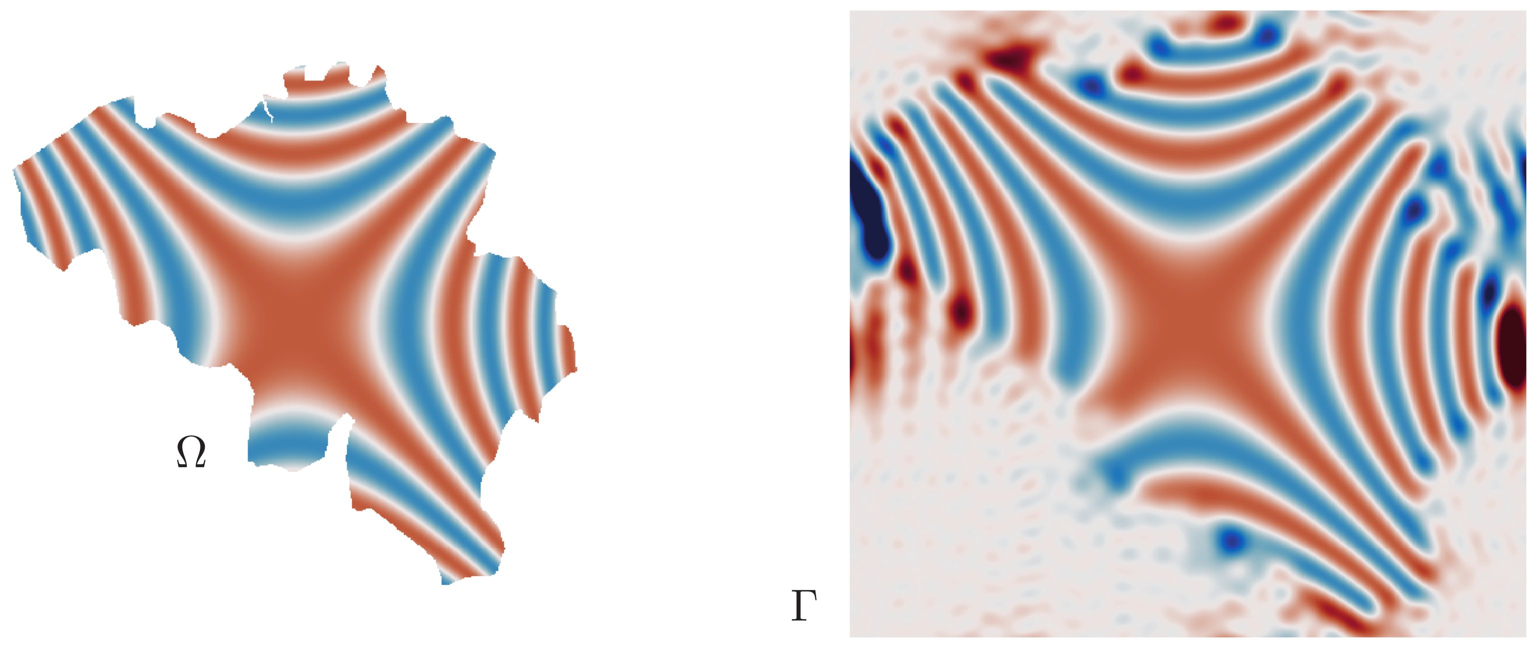
Challenges: The redundancy of a frame can lead to extremely ill-conditioned linear systems, but despite this, *stable and fast algorithms are possible* if we proceed with care.

Examples of frames

Fourier extensions: For a domain $\Omega \subset \Gamma$, where $\Gamma = [-1,1]^d$ or some other bounding box, use the functions which are the Fourier series on Γ, but restricted to Ω:

$$\Phi = \{\exp(i\pi \mathbf{k}\cdot\mathbf{x}) : \mathbf{k} \in \mathbb{Z}^d\} \subset L^2(\Omega).$$

Particularly powerful if Ω has complicated geometry.



Augmented Fourier basis: add a finite number of polynomials to the Fourier basis:

$$\Phi = \{\exp(i\pi k x) : k \in \mathbb{Z}\} \cup \{P_1(x), P_2(x), \ldots, P_d(x)\}.$$

This can reduce the Gibbs phenomenon for nonperiodic functions.

Polynomial plus modified polynomials: Take a polynomial basis plus $w(x)$ times polynomial basis:

$$\Phi = \{P_0(x), P_1(x), \ldots\} \cup \{w(x)P_0(x), w(x)P_1(x), \ldots\}.$$

The weight function w can be complex, and may be singular, oscillatory or possess some other feature which makes classical approximation difficult.

Best approximation

- Given a frame Φ and a finite subset Φ_N, compute the orthogonal projection $\mathcal{P}_N f$ of f onto $\mathcal{H}_N = \mathrm{span}(\Phi_N)$.
- This can be done by solving the linear system, $G_N x = b$, where

$$(G_N)_{k,j} = \langle \phi_k, \phi_j\rangle, \qquad b_k = \langle \phi_k, f\rangle.$$

$$\mathcal{P}_N f = \sum_{\sigma_k > 0} \frac{\langle \phi_k, f\rangle}{\sigma_k}\phi_k, \qquad \sigma(G_N) = \{\sigma_1, \ldots, \sigma_N\}.$$

- If Φ is an orthonormal basis, this is trivial, since $G_N = I_N$. In general, however, G_N can be arbitrarily badly conditioned. Furthermore, the solution's norm $\|x\|_2$ can grow arbitrarily rapidly as N increases.
- This means that in general, it is effectively impossible to compute the best approximation.

Stable computation

Since computation of $\mathcal{P}_N f$ is in general practically impossible, we instead compute the ε-truncation,

$$\mathcal{P}_N^\varepsilon f = \sum_{\sigma_k > \varepsilon} \frac{\langle \phi_k, f\rangle}{\sigma_k}\phi_k, \qquad \sigma(G_N) = \{\sigma_1, \ldots, \sigma_N\}.$$

For even greater stability, an $M \times N$ system can be solved ($M \geq N$, so solved in the least squares sense), $G_{M,N}x = b$, where

$$(G_{M,N})_{k,j} = \langle \phi_k, \phi_j\rangle,$$

$$\mathcal{P}_{M,N}^\varepsilon f = \sum_{\sigma_k > \varepsilon} \frac{\langle \phi_k, f\rangle}{\sigma_k}\phi_k, \qquad s(G_{M,N}) = \{\sigma_1, \ldots, \sigma_N\}.$$

Convergence of truncated projections

Theorem 1. The truncated SVD projection $\mathcal{P}_N^\varepsilon$ satisfies

$$\|f - \mathcal{P}_N^\varepsilon f\| \leq \|f - \mathcal{T}_N z\| + \sqrt{\varepsilon}\|z\|, \qquad \forall z \in \mathbb{C}^N, f \in \mathcal{H}.$$

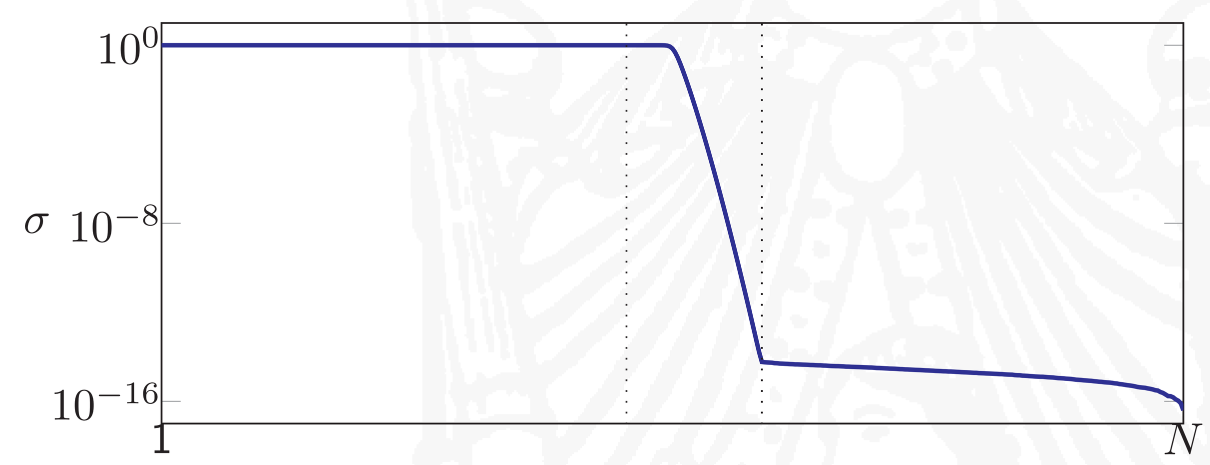
Theorem 2. The oversampled, truncated SVD projection $\mathcal{P}_{M,N}^\varepsilon$ satisfies

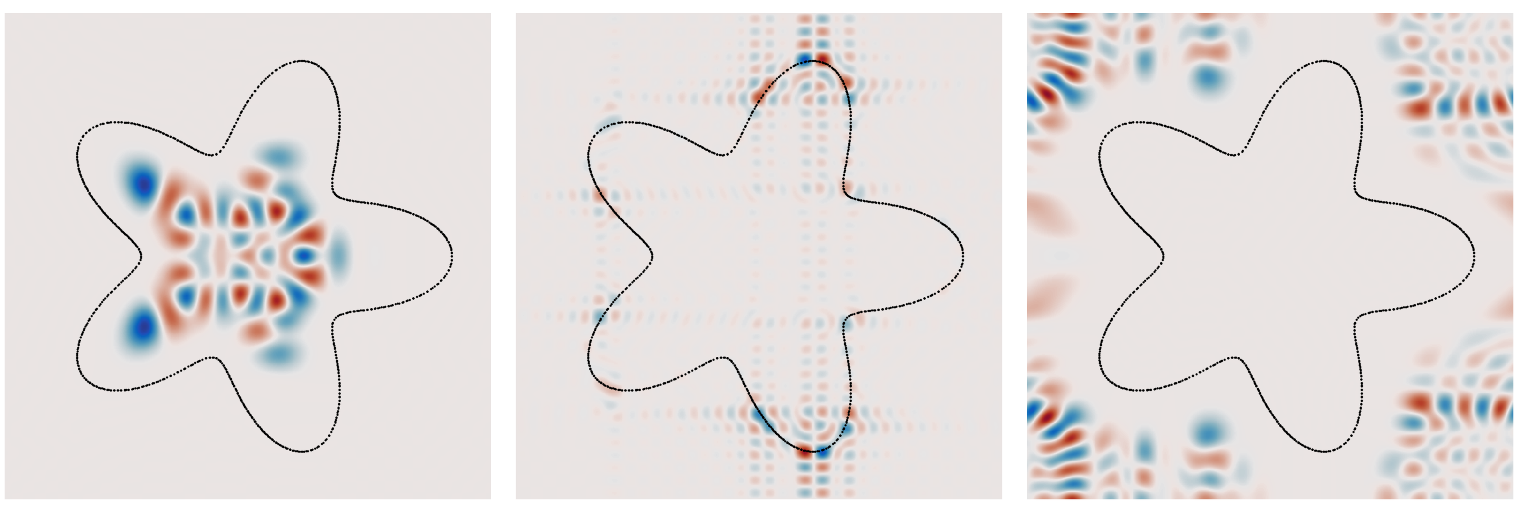
$$\limsup_{M\to\infty} \|f - \mathcal{P}_{M,N}^\varepsilon f\| \leq C(\|f - \mathcal{T}_N z\| + \varepsilon\|z\|), \qquad \forall z \in \mathbb{C}^N, f \in \mathcal{H}.$$

Important property: These two theorems show that the computation is stable if there exists some vector of coefficients z which gives a good approximation of f and such that $\|z\|$ is small.

A fast algorithm for Fourier extensions

The plunge region: For Fourier extensions, $G_{M,N}$ has a distinctive spectrum with three parts: (i) $O(N)$ singular values are close to 1, corresponding to functions concentrated in the interior of Ω. (ii) A plunge region where $1-\varepsilon > \sigma > \varepsilon$, corresponding to the boundary. This grows as $o(N)$. (iii) The region where $\sigma \leq \varepsilon$, these are truncated.





$\sigma \sim 1-\varepsilon$ $\qquad$ $\sigma \sim 0.5$ $\qquad$ $\sigma \sim \varepsilon$

Fast Algorithm: The related matrix $(I - G_{M,N}G_{M,N}^*)G_{M,N}$ isolates the plunge region. This *low-rank* problem can be solved using a fast randomized SVD algorithm. What remains vanishes at the boundary and is solved through regular FFTs.

References

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